The Resolvent Algebra: A New Approach to Canonical Quantum Systems

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Abstract

The standard C*–algebraic version of the algebra of canonical commutation relations, the Weyl algebra, frequently causes difficulties in applications since it neither admits the formulation of physically interesting dynamical laws nor does it incorporate pertinent physical observables such as (bounded functions of) the Hamiltonian. Here a novel C*–algebra of the canonical commutation relations is presented which does not suffer from such problems. It is based on the resolvents of the canonical operators and their algebraic relations. The resulting C*–algebra, the resolvent algebra, is shown to have many desirable analytic properties and the regularity structure of its representations is surprisingly simple. Moreover, the resolvent algebra is a convenient framework for applications to interacting and to constrained quantum systems, as we demonstrate by several examples.

1 Introduction

Canonical systems of operators have always been a central ingredient in the modelling of quantum systems. There is an extensive literature analyzing their properties, starting with the seminal paper of Born, Jordan and Heisenberg on the physical foundations and reaching a first mathematical satisfactory formulation in the works of von Neumann and of Weyl. These canonical systems of operators may all be presented in the following general form: there is a real linear map ϕ from a given symplectic space (X, σ) to a linear space of essentially selfadjoint operators on some common dense invariant core $\mathcal{D}$ in a Hilbert space $\mathcal{H}$, satisfying the relations

$$[\phi(f), \phi(g)] = i\sigma(f, g)\, \mathbb{1}, \quad \phi(f)^* = \phi(f) \quad \text{on} \quad \mathcal{D}\,.$$

In the case that X is finite dimensional, we can reinterpret this relation in terms of the familiar quantum mechanical position and momentum operators, and if X consists of Schwartz functions on a space–time manifold one may consider ϕ to be a bosonic quantum field. The observables of the system are then constructed from the operators $\{\phi(f) \mid f \in X\}$, usually as polynomial expressions. Since one wants to study representations of such systems, some care needs to be taken about the appropriate mathematical framework to do this in, especially since there are known pathologies *e.g.* for the case that X is infinite dimensional. Here we will use C*-algebras to

encode the algebraic information of the canonical systems, given the rich source of mathematical tools available.

As is well–known, if $(X,\,\sigma)$ is non–degenerate then the operators $\phi(f)$ cannot all be bounded. Thus starting from the polynomial algebra $\mathcal{P}$ generated by $\{\phi(f)\,|\,f\in X\}$ we have to obtain a C*–algebra encoding the same algebraic information, necessarily in bounded form. The obvious way to take this step is to form suitable bounded functions of the fields $\phi(f)$. In the approach introduced by Weyl, this is done by considering the C*–algebra generated by the set of unitaries

$$\{\,\exp\left(i\phi(f)\right)\,|\,f\in X\}$$

and this C*–algebra is simple. It can be defined abstractly as the C*–algebra generated by a set of unitaries $\{\,\delta_f\;|\;f\in X\,\}$ subject to the relations $\delta_f^*=\delta_{-f}$ and $\delta_f\delta_g=e^{-i\sigma(f,g)/2}\delta_{f+g}$. This is the familiar Weyl (or CCR) algebra, often denoted $\overline{\Delta(X,\,\sigma)}$ [20]. By its definition, it has a representation in which the unitaries δ_f can be identified with the exponentials $e^{i\phi(f)}$, and hence we can obtain the concrete algebra $\mathcal{P}$ back from these. Such representations $\pi:\overline{\Delta(X,\,\sigma)}\to\mathcal{B}(\mathcal{H})$, *i.e.* those for which the one–parameter groups $\lambda\to\pi(\delta_{\lambda f})$ are strong operator continuous for all $f\in X$ are called regular. Since for physical situations the quantum fields are defined as the generators of the one–parameter groups $\lambda\to\pi(\delta_{\lambda f})$, the representations of interest are required to be regular.

The Weyl algebra suffers from several well–known flaws. First and foremost, the dynamics (one–parameter automorphism groups) of the Weyl algebra which are most naturally defined from symplectic transformations of X, correspond to the dynamics produced by quadratic Hamiltonians, and this excludes most physically interesting situations. As a matter of fact, in the case $X=\mathbb{R}^2$ in the Schrödinger representation for $\overline{\Delta(X,\,\sigma)}$, the time evolutions obtained from Hamiltonians of the form $H=P^2+V(Q)$ for potentials $V\in L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$ do not preserve $\overline{\Delta(X,\,\sigma)}$ unless $V=0$ (cf. [10]). Thus the Weyl algebra does not allow the definition of much interesting dynamics on it, and this limits its usefulness. Second, in regular representations, natural observables such as bounded functions of the Hamiltonian are not in $\overline{\Delta(X,\,\sigma)}$. Third, the Weyl algebra has a large number of nonregular representations [1,14,15]. Through the use of w*–limits of regular states these are interpreted as situations where the field $\phi(f)$ can have "infinite field strength". Whilst this is useful for some idealizations e.g. plane waves cf. [1] or for quantum constraints cf. [15], for most physical situations one wants to exclude such representations.

This motivates the consideration of alternative versions of the C*–algebra of canonical commutation relations. Instead of taking the C*–algebra generated by exponentials of the underlying fields, as for the Weyl algebra, we will consider the C*–algebra generated by the resolvents of the fields. These are formally given by $R(\lambda,f):=(i\lambda\mathbb{1}-\phi(f))^{-1}$. All algebraic properties of the fields can be expressed in terms of simple relations amongst these resolvents. The unital C*-algebra generated by the resolvents,

$$\mathcal{R}:=\mathrm{C}^*\{\,R(\lambda,f)\;|\;f\in X,\,\lambda\in\mathbb{R}\backslash 0\}\,,$$

called the resolvent algebra, will be the subject of our investigation, and below we will exhibit its main algebraic and analytic properties.

As a preview, consider for fixed $f \in X$ the abelian subalgebra $\mathcal{R}_f := \mathrm{C}^*\{ R(\lambda, f) \mid \lambda \in \mathbb{R}\}$. This algebra is isomorphic to the algebra of continuous functions on $\mathbb{R}$, vanishing at infinity. Thus all states of "infinite field strength" for $\phi(f)$ annihilate this algebra, which greatly simplifies the representation theory of $\mathcal{R}$. The obvious price to pay is that $\mathcal{R}$ is not simple; but this non–trivial ideal structure turns out to be useful in applications. Moreover, in contrast to the Weyl algebra, the underlying unbounded field operators are affiliated with the C*–algebra $\mathcal{R}$ in the sense of Damak and Georgescu [6]. This feature allows one to manipulate rigorously polynomial expressions of the fields by the use of "mollifiers" (explained below). As a matter of fact, the latter observation motivated us to introduce the resolvent algebra in our recent study of supersymmetry [5], where we needed these mollifiers to construct superderivations. Thus, in many respects, the resolvent algebra provides a technically convenient framework for the study of canonical quantum systems.

The structure of this paper is as follows. In Sect. 2 we introduce the notion of mollifiers to encode polynomials in the bosonic fields in bounded form, and we show that the Weyl algebra does not have any mollifiers, whereas the resolvent algebra does. In Sect. 3 we define the resolvent algebra abstractly as a C*–algebra, and establish basic algebraic properties. In Sect. 4 we study states and representations of the resolvent algebra, in particular we introduce the notion of a "regular" representation and show that these are in a natural bijection with the usual regular representations of the Weyl algebra. We obtain interesting decompositions of the symplectic space associated to representations, and we prove that every regular representation is faithful. In Sect. 5 we consider further algebraic properties and find that the resolvent algebra is nonseparable, find a tensor product structure for some of its subalgebras, and for the case of a finite dimensional symplectic space, find that it contains a copy of the algebra of compact operators. In Sect. 6 we consider how to encode dynamics and Hamiltonians in the resolvent algebra. Many resolvents of Hamiltonians are already in the resolvent algebra, and using the copies of the compacts in the resolvent algebra, we can encode many more dynamical systems than for the Weyl algebra. We then produce two applications to illustrate the usefulness of the resolvent algebra. In Sect. 7 we develop a model of an infinite family of atoms which are confined around the points of a lattice by a pinning potential and interact with their nearest neighbours. We construct a ground state for it by algebraic means. In Sect. 8 we take a brief look at Dirac constraints theory for linear bosonic constraints in the context of the resolvent algebra, and find that it is considerably simpler than in the Weyl algebra. All proofs for our results are collected in Sect. 10.

2 Mollifiers and Resolvents

There are several concepts of when a selfadjoint operator A on a Hilbert space $\mathcal{H}$ is *affiliated* with a concretely represented C*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$. One is that the resolvent $(i\lambda \mathbb{1} - A)^{-1} \in \mathcal{A}$ for some $\lambda \in \mathbb{R}\backslash 0$ (hence for all $\lambda \in \mathbb{R}\backslash 0$). This notion is used by Damak and Georgescu [6] (and is weaker than the one used by Woronowicz [29]) and it implies the usual one, *i.e.* that A commutes

with all unitaries commuting with $\mathcal{A}$ (but not conversely). Observe that then

$$A(i\lambda\mathbb{1}-A)^{-1}=\overline{(i\lambda\mathbb{1}-A)^{-1}A}=i\lambda(i\lambda\mathbb{1}-A)^{-1}-\mathbb{1}\in\mathcal{A}\,.$$

Thus the resolvent $(i\lambda\mathbb{1}-A)^{-1}=M$ acts as a "mollifier" for A, *i.e.* $\overline{MA}$ and AM are bounded and in $\mathcal{A}$, and M is invertible such that $M^{-1}MA=A=AMM^{-1}$. This suggests that as AM and $\overline{MA}$ in $\mathcal{A}$ carries the information of A in bounded form, we can "forget" the original representation, and study the affiliated A abstractly through these elements. In the literature, the bounded operators $A_\lambda:=i\lambda A\,(i\lambda\mathbb{1}-A)^{-1}$ are called "Yosida approximations" of A [22, p 9].

We want to apply this idea to a bosonic field as above, *i.e.* for a fixed Hilbert space $\mathcal{H}$ we assume that there is a common dense invariant core $\mathcal{D}\subset\mathcal{H}$ for the selfadjoint operators $\phi(f)$, $f\in X$ on which the $\phi(f)$ satisfy the canonical commutation relations. One may be tempted to find mollifiers for the operators $\phi(f)$ in the (concretely represented) Weyl algebra

$$\overline{\Delta(X,\,\sigma)}=C^*\{\exp(i\phi(f))\;\mid\;f\in X\}\subset\mathcal{B}(\mathcal{H})\,,$$

but unfortunately this is not possible because [5]:

2.1 Proposition *The Weyl algebra $\overline{\Delta(X,\,\sigma)}$ contains no nonzero element M such that $\phi(f)M$ is bounded for some $f\in X\backslash 0$. Thus $\overline{\Delta(X,\,\sigma)}$ contains no mollifier for any nonzero $\phi(f)$, and $\phi(f)$ is not affiliated with $\overline{\Delta(X,\,\sigma)}$.*

Our solution is to abandon the Weyl algebra as the appropriate C*-algebra to model the bosonic fields $\phi(f)$, and instead to choose the unital C*-algebra generated by the resolvents,

$$C^*\{\,R(\lambda,f)\;\mid\;\lambda\in\mathbb{R}\backslash 0,\;f\in X\,\}\;,$$

where $R(\lambda,f):=(i\lambda\mathbb{1}-\phi(f))^{-1}$. Then by construction all $\phi(f)$ are affiliated to this C*-algebra and it contains mollifiers $R(\lambda,f)$ for all of them.

3 Resolvent Algebra - Basics

The above discussion took place in a concrete setting, *i.e.* represented on a Hilbert space, and we would like to abstract this. Just as the Weyl algebra can be abstractly defined by the Weyl relations, we want to abstractly define the C*-algebra of resolvents by its generators and relations.

3.1 Definition *Given a symplectic space $(X,\,\sigma)$, we define $\mathcal{R}_0$ to be the universal unital *-algebra generated by the set $\{\,R(\lambda,f)\;\mid\;\lambda\in\mathbb{R}\backslash 0,\;f\in X\,\}$ and the relations*

$$R(\lambda,0)=-\frac{i}{\lambda}\mathbb{1}\tag{1}$$

$$R(\lambda,f)^*=R(-\lambda,f)\tag{2}$$

$$\nu\,R(\nu\lambda,\,\nu f)=R(\lambda,f)\tag{3}$$

$$R(\lambda,f)-R(\mu,f)=i(\mu-\lambda)R(\lambda,f)R(\mu,f)\tag{4}$$

$$[R(\lambda,f),\,R(\mu,g)]=i\sigma(f,g)\,R(\lambda,f)\,R(\mu,g)^2R(\lambda,f)\tag{5}$$

$$R(\lambda,f)R(\mu,g)=R(\lambda+\mu,\,f+g)[R(\lambda,f)+R(\mu,g)+i\sigma(f,g)R(\lambda,f)^2R(\mu,g)]\tag{6}$$

*where λ, μ, $\nu \in \mathbb{R}\backslash 0$ and f, $g \in X$, and for (6) we require $\lambda + \mu \neq 0$. That is, start with the free unital *-algebra generated by $\{ R(\lambda, f) \mid \lambda \in \mathbb{R}\backslash 0,\ f \in X \}$ and factor out by the ideal generated by the relations (1) to (6) to obtain the *-algebra $\mathcal{R}_0$.*

3.2 Remarks (a) The *-algebra $\mathcal{R}_0$ is nontrivial, because it has nontrivial representations. For instance, in a Fock representation π of the CCRs over (X, σ) one has the selfadjoint CCR-fields $\phi_\pi(f)$, $f \in X$ from which one can define $\pi(R(\lambda, f)) := (i\lambda\mathbb{1} - \phi_\pi(f))^{-1}$ to obtain a representation of $\mathcal{R}_0$.

(b) Obviously (2) encodes the selfadjointness of $\phi_\pi(f)$, (3) encodes $\phi_\pi(\nu f) = \nu\phi_\pi(f)$, (4) encodes that $R(\lambda, f)$ is a resolvent, (5) encodes the canonical commutation relations and (6) encodes additivity $\phi_\pi(f+g) = \phi_\pi(f) + \phi_\pi(g)$. Note that $R(0, f)$ is undefined.

(c) Let $\mu = -\lambda$ in Equation (4) to get the useful equation

$$R(\lambda, f) - R(\lambda, f)^* = -2i\lambda R(\lambda, f)R(\lambda, f)^* \,. \tag{7}$$

To define our resolvent C*-algebra, we need to decide on which C*-seminorm to define on $\mathcal{R}_0$. The obvious choice is the enveloping C*-seminorm, however since $\mathcal{R}_0$ is merely a *-algebra, we need to establish some uniform boundedness for its Hilbert space representations before we can define this.

3.3 Proposition *Given a symplectic space (X, σ), define $\mathcal{R}_0$ as above.*

*(i) Let $\pi_0 : \mathcal{R}_0 \to \mathcal{B}(\mathcal{H})$ be a *-representation of $\mathcal{R}_0$, where $\mathcal{H}$ is a Hilbert space. Then $\|\pi_0(R(\lambda, f))\| \leq |\lambda|^{-1}$. Thus, for each $A \in \mathcal{R}_0$ there is a $c_A \geq 0$ such that $\|\pi(A)\| \leq c_A$ for all (bounded) Hilbert space representations π of $\mathcal{R}_0$.*

*(ii) Let ω be a positive functional of $\mathcal{R}_0$, i.e. $\omega : \mathcal{R}_0 \to \mathbb{C}$ is linear and $\omega(A^*A) \geq 0$ for all $A \in \mathcal{R}_0$. Then the GNS–construction yields a cyclic *–representation of $\mathcal{R}_0$, denoted π_ω, consisting of bounded Hilbert space operators.*

Thus by (i) we can form direct sum representations over infinite sets of representations and still maintain boundedness. However the class of (nondegenerate) representations is not a set[1], so we need to find a suitable set of representations. Let $\mathfrak{S}$ denote the set of positive functionals ω of $\mathcal{R}_0$ for which $\omega(\mathbb{1}) = 1$, then by Proposition 3.3(ii) their GNS–representations are bounded, and in fact by (i) they are uniformly bounded. Since $\mathfrak{S}$ is a set, we can now sensibly define:

3.4 Definition *The universal representation $\pi_u : \mathcal{R}_0 \to \mathcal{B}(\mathcal{H}_u)$ is given by*

$$\pi_u(A) := \bigoplus \{\pi_\omega(A) \,|\, \omega \in \mathfrak{S}\} \quad \textit{and} \quad \|A\|_u := \|\pi_u(A)\| = \sup_{\omega \in \mathfrak{S}} \|\pi_\omega(A)\|$$

denotes the enveloping C–seminorm of $\mathcal{R}_0$; note that $\|A\|_u = \sup_{\omega \in \mathfrak{S}} \omega(A^*A)^{1/2}$ since $\mathfrak{S}$ contains all vector states of all its GNS–representations. We define our resolvent algebra $\mathcal{R}(X, \sigma)$ as the abstract C*-algebra generated by $\pi_u(\mathcal{R}_0)$, i.e. we factor $\mathcal{R}_0$ by $\mathrm{Ker}\,\pi_u$ and complete w.r.t. the enveloping C*–seminorm $\|\cdot\|_u$.*

[1] If the nondegenerate representations were a set, we could take the direct sum of the representations which do not have themselves as a direct summand, to obtain Russell's paradox.

3.5 Remark Previously, in [5] we defined the resolvent algebra with a different C*–seminorm, because we needed more analytic structure. Henceforth we will deal with the resolvent algebra $\mathcal{R}(X,\,\sigma)$ as defined above and denote its norm by $\|\cdot\|$. Below we will prove isomorphism with the previous version.

We state some elementary properties of $\mathcal{R}(X,\,\sigma)$.

3.6 Theorem *Let $(X,\,\sigma)$ be a given nondegenerate symplectic space, and define $\mathcal{R}(X,\,\sigma)$ as above. Then for all $\lambda,\,\mu\in\mathbb{R}\backslash 0$ and $f,\,g\in X$ we have:*

(i) $[R(\lambda,f),\,R(\mu,f)]=0$. *Substitute $\mu=-\lambda$ to see that $R(\lambda,f)$ is normal.*

(ii) $R(\lambda,f)R(\mu,g)^2R(\lambda,f)=R(\mu,g)R(\lambda,f)^2R(\mu,g)$.

(iii) $\|R(\lambda,f)\|=|\lambda|^{-1}$.

(iv) *$R(\lambda,f)$ is analytic in λ. Explicitly, the series expansion (von Neumann series)*

$$R(\lambda,f)=\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n\,i^nR(\lambda_0,f)^{n+1},\qquad \lambda,\;\lambda_0\neq 0$$

converges absolutely in norm whenever $|\lambda_0-\lambda|<|\lambda_0|$.

(v) *Let $T\in\mathrm{Sp}(X,\sigma)$ be a symplectic transformation. Then $\alpha(R(\lambda,f)):=R(\lambda,Tf)$ extends to an automorphism $\alpha\in\mathrm{Aut}\,\mathcal{R}(X,\,\sigma)$.*

Note that the von Neumann series for $R(\lambda,f)$ converges for any $\lambda\in\mathbb{C}$ with $|\lambda-\lambda_0|<|\lambda_0|$, *i.e.* on a disk which stays off the imaginary axis. Using different $\lambda_0\in\mathbb{R}\backslash 0$ we can thus define $R(z,f)$ for any complex z not on the imaginary axis, *i.e.* analytically continue $R(\lambda,f)$ from $\mathbb{R}\backslash 0$ to $\mathbb{C}\backslash i\mathbb{R}$. Thus $\mathcal{R}(X,\,\sigma)$ contains also 'resolvents' $R(z,f)$ for complex $z\in\mathbb{C}\backslash i\mathbb{R}$. In fact, we can define $\mathcal{R}(X,\,\sigma)$ as the enveloping C*–algebra of the universal unital *-algebra generated by the set $\{\,R(z,f)\;\mid\;z\in\mathbb{C}\backslash i\mathbb{R},\;f\in X\,\}$ and the analytic continuations of the relations (1)–(6) *i.e.*

$$\begin{aligned}
R(z,0) &= -\frac{i}{z}\,\mathbb{1} && (8)\\
R(z,f)^* &= R(-\overline{z},f) && (9)\\
\nu R(\nu z,\nu f) &= R(z,\,f)\,,\quad \nu\in\mathbb{R}\backslash 0 && (10)\\
R(z,f)-R(w,f) &= i(z-w)R(z,f)\,R(w,f) && (11)\\
[R(z,f),\,R(w,g)] &= i\sigma(f,g)\,R(z,f)\,R(w,g)^2R(z,f) && (12)\\
R(z,f)R(w,g) &= R(z+w,\,f+g)[R(z,f)+R(w,g)+i\sigma(f,g)R(z,f)^2R(w,g)] && (13)
\end{aligned}$$

where $z,\,w\in\mathbb{C}\backslash i\mathbb{R}$ and $f,\,g\in X$, and for (13) we require $z+w\notin i\mathbb{R}$. Note that Eq. (7) becomes

$$R(z,f)-R(z,f)^*=i(z+\overline{z})R(z,f)R(z,f)^*\,, \tag{14}$$

hence we get $\|R(z,f)\|=|\mathrm{Re}\,z|^{-1}$. Since by Eq. (11) we can again (via a von Neumann series) prove analyticity off the imaginary axis, it follows that this C*–algebra coincides with the previously defined $\mathcal{R}(X,\,\sigma)$. Making use of this fact, we can exhibit another family of automorphisms of the resolvent algebra which will be useful for proofs below.

3.7 Proposition *Let $(X,\,\sigma)$ be a given nondegenerate symplectic space. Then for each linear map $h : X \to \mathbb{R}$ there is an automorphism $\beta_h \in \operatorname{Aut}\mathcal{R}(X,\,\sigma)$ defined by*

$$\beta_h(R(z,f)) = R(z+ih(f),\,f)$$

for $f \in X$ and $z \in \mathbb{C}\backslash i\mathbb{R}$.

As already stated, the resolvent algebra is not simple, in contrast to the Weyl algebra. More specifically:

3.8 Theorem *Let $(X,\,\sigma)$ be a given nondegenerate symplectic space. Then for all $\lambda \in \mathbb{R}\backslash 0$ and $f \in X\backslash 0$ we have that the closed two–sided ideal generated by $R(\lambda, f)$ in $\mathcal{R}(X,\,\sigma)$ is*

$$[R(\lambda,f)\mathcal{R}(X,\sigma)] = [\mathcal{R}(X,\sigma)R(\lambda,f)] = [\mathcal{R}(X,\sigma)R(\lambda,f)\mathcal{R}(X,\sigma)]$$

where $[\,\cdot\,]$ indicates the closed linear span of its argument. This ideal is proper. Moreover the intersection of the ideals $[R(\lambda_i,f_i)\mathcal{R}(X,\sigma)]$, $i = 1,\ldots,n$ for distinct $f_i \in X\backslash 0$ is the ideal $[R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\mathcal{R}(X,\sigma)]$.

From these ideals we can build other ideals, *e.g.* for a set $S \subseteq X$ we can define the ideals $\bigcap\limits_{f\in S}[\mathcal{R}(X,\sigma)R(\lambda,f)]$ as well as $\Big[\bigcup\limits_{f\in S}[\mathcal{R}(X,\sigma)R(\lambda,f)]\Big]$. Ideals of a different structure will occur in the following sections. Thus $\mathcal{R}(X,\sigma)$ has a very rich ideal structure.

The fact that $\mathcal{R}(X,\sigma)$ is not simple, does not disqualify it from being used as the observable algebra of a physical system, because we will show below that its images in all physical (*i.e.* regular) representations are isomorphic.

4 States, representations and regularity

Any operator family R_λ, $\lambda \in \mathbb{R}\backslash 0$ on some Hilbert space satisfying the resolvent equation (4) is called by Hille a pseudo-resolvent and for such a family we know (cf. Theorem 1 in [30, p 216]) that:

- All R_λ have a common range and a common null space.
- A pseudo-resolvent is the resolvent for an operator B iff $\operatorname{Ker} R_\lambda = \{0\}$ for some (hence for all) $\lambda \in \mathbb{R}\backslash 0$, and in this case $\operatorname{Dom} B = \operatorname{Ran} R_\lambda$ for all $\lambda \in \mathbb{R}\backslash 0$.

This leads us to an examination of $\operatorname{Ker}\pi(R(\lambda,f))$ for representations π.

4.1 Theorem *Let $(X,\,\sigma)$ be a given nondegenerate symplectic space, and define $\mathcal{R}(X,\,\sigma)$ as above. Then for $\lambda \in \mathbb{R}\backslash 0$ and $f \in X\backslash 0$ we have:*

(i) If for a representation π of $\mathcal{R}(X,\sigma)$ we have $\operatorname{Ker}\pi(R(\lambda,f)) \neq \{0\}$, then $\operatorname{Ker}\pi(R(\lambda,f))$ reduces $\pi(\mathcal{R}(X,\sigma))$. Hence there is a unique orthogonal decomposition $\pi = \pi_1 \oplus \pi_2$ such that $\pi_1(R(\lambda,f)) = 0$ and $\pi_2(R(\lambda,f))$ is invertible.

(ii) *Let π be any nondegenerate representation of $\mathcal{R}(X,\sigma)$, then*

$$P_f := \operatorname*{s-lim}_{\lambda\to\infty} i\lambda\,\pi(R(\lambda,f))$$

exists, defines a central projection of $\pi(\mathcal{R}(X,\sigma))''$, and it is the range projection of $\pi(R(\lambda,f))$ as well as the projection of the ideal $\pi\left([\mathcal{R}(X,\sigma)R(\lambda,f)]\right)$.

(iii) *If π is a factorial representation of $\mathcal{R}(X,\sigma)$, then $P_f = 0$ or $\mathbb{1}$ and such π are classified by the sets $\{f \in X\backslash 0 \mid P_f = \mathbb{1}\}$.*

(iv) *There is a state $\omega \in \mathfrak{S}(\mathcal{R}(X,\sigma))$ such that $R(\lambda,f) \in \operatorname{Ker}\omega$. Moreover, given a state ω with $R(\lambda,f) \in \operatorname{Ker}\omega$, then $R(\lambda,f) \in \operatorname{Ker}\pi_\omega$.*

Given a $\pi \in \operatorname{Rep}(\mathcal{R}(X,\,\sigma),\mathcal{H}_\pi)$ with $\operatorname{Ker}\pi(R(1,f)) = \{0\}$, we can define a field operator by

$$\phi_\pi(f) := i\mathbb{1} - \pi(R(1,f))^{-1}$$

with domain $\operatorname{Dom}\phi_\pi(f) = \operatorname{Ran}\pi(R(1,f))$, and it has the following properties:

4.2 Theorem *Let $\mathcal{R}(X,\,\sigma)$ be as above, and let $\pi \in \operatorname{Rep}(\mathcal{R}(X,\,\sigma),\mathcal{H}_\pi)$ satisfy $\operatorname{Ker}\pi(R(1,f)) = \{0\} = \operatorname{Ker}\pi(R(1,h))$ for given $f,\ h \in X$. Then*

(i) *$\phi_\pi(f)$ is selfadjoint, and $\pi(R(\lambda,f))\operatorname{Dom}\phi_\pi(h) \subseteq \operatorname{Dom}\phi_\pi(h)$.*

(ii) $\lim\limits_{\lambda\to\infty} i\lambda\pi(R(\lambda,f))\Psi = \Psi$ *for all $\Psi \in \mathcal{H}_\pi$.*

(iii) $\lim\limits_{\mu\to 0} i\pi(R(1,\mu f))\Psi = \Psi$ *for all $\Psi \in \mathcal{H}_\pi$.*

(iv) *The space $\mathcal{D} := \pi(R(1,f)R(1,h))\mathcal{H}_\pi$ is a joint dense domain for $\phi_\pi(f)$ and $\phi_\pi(h)$ and we have: $[\phi_\pi(f),\,\phi_\pi(h)] = i\sigma(f,h)\mathbb{1}$ on $\mathcal{D}$.*

(v) $\operatorname{Ker}\pi(R(1,\nu f + h)) = \{0\}$ *for $\nu \in \mathbb{R}$. Then $\phi_\pi(\nu f + h)$ is defined, $\mathcal{D}$ is a core for $\phi_\pi(\nu f + h)$ and $\phi_\pi(\nu f + h) = \nu\phi_\pi(f) + \phi_\pi(h)$ on $\mathcal{D}$. Moreover $\pi(R(1,\nu f + h)) \in \{\pi(R(1,f)),\,\pi(R(1,h))\}''$.*

(vi) *$\phi_\pi(f)\pi(R(\lambda,f)) = \pi(R(\lambda,f))\phi_\pi(f) = i\lambda\pi(R(\lambda,f)) - \mathbb{1}$ on $\operatorname{Dom}\phi_\pi(f)$.*

(vii) *$[\phi_\pi(f),\pi(R(\lambda,h))] = i\sigma(f,h)\pi(R(\lambda,h)^2)$ on $\operatorname{Dom}\phi_\pi(f)$.*

(viii) *Denote $W(f) := \exp(i\phi_\pi(f))$, then*

$$\begin{aligned} W(f)W(h) &= e^{-i\sigma(f,h)/2}\,W(f+h) \\ W(f)\pi(R(\lambda,h))W(f)^{-1} &= \pi(R(\lambda + i\sigma(h,f),\,h)) = \pi(\beta_{f_\sigma}(R(\lambda,h))) \end{aligned}$$

where $f_\sigma : X \to \mathbb{R}$ is given by $f_\sigma(h) := \sigma(h,f)$, $h \in X$. Moreover $W(sf)\mathcal{D} \subseteq \mathcal{D} \supseteq W(th)\mathcal{D}$ for $s,\,t \in \mathbb{R}$, hence $\mathcal{D}$ is a common core for $\phi_\pi(f)$ and $\phi_\pi(h)$.

Thus we define:

4.3 Definition *A representation* $\pi \in \mathrm{Rep}\,(\mathcal{R}(X,\,\sigma),\mathcal{H}_\pi)$ *is regular on a set* $S \subset X$ *if*

$$\mathrm{Ker}\,\pi(R(1,f)) = \{0\} \qquad \textit{for all } f \in S\,.$$

A state ω *of* $\mathcal{R}(X,\sigma)$ *is regular on a set* $S \subset X$ *if its GNS–representation* π_ω *is regular on* $S \subset X$*. A regular representation (resp. state) is a representation (resp. state) which is regular on* X*. Given a Hilbert space* $\mathcal{H}$*, we denote the set of (nondegenerate) regular representations* $\pi : \mathcal{R}(X,\sigma) \to \mathcal{B}(\mathcal{H})$ *by* $\mathrm{Reg}(\mathcal{R}(X,\sigma),\mathcal{H})$*. The set of regular states of* $\mathcal{R}(X,\sigma)$ *is denoted by* $\mathfrak{S}_r(\mathcal{R}(X,\sigma))$*.*

Obviously many regular representations are known, *e.g.* the Fock representation. Recall that the class of all regular representations of $\mathcal{R}(X,\sigma)$ is not a set, hence the necessity to fix $\mathcal{H}$. Thus for $\pi \in \mathrm{Reg}(\mathcal{R}(X,\sigma),\mathcal{H})$, all the field operators $\phi_\pi(f)$, $f \in X$ are defined, and we have the resolvents $\pi(R(\lambda,f)) = (i\lambda\mathbb{1} - \phi_\pi(f))^{-1}$.

From Theorem 4.2, we can now establish a bijection between the regular representations of $\mathcal{R}(X,\sigma)$ and the regular representations of the Weyl algebra $\overline{\Delta(X,\,\sigma)}$:

4.4 Corollary *Let* $\mathcal{R}(X,\sigma)$ *be as above. Given a regular representation* $\pi \in \mathrm{Reg}(\mathcal{R}(X,\sigma),\mathcal{H})$*, define a regular representation* $\widetilde{\pi} \in \mathrm{Reg}(\overline{\Delta(X,\,\sigma)},\mathcal{H})$ *by* $\widetilde{\pi}(\delta_f) := \exp(i\phi_\pi(f)) = W(f)$ *(using Theorem 4.2(viii)). This correspondence establishes a bijection between* $\mathrm{Reg}(\mathcal{R}(X,\sigma),\mathcal{H})$ *and* $\mathrm{Reg}(\overline{\Delta(X,\,\sigma)},\mathcal{H})$ *which respects irreducibility and direct sums. Its inverse is given by the Laplace transform,*

$$\pi(R(\lambda,f)) := -i\int_0^{\sigma\infty} e^{-\lambda t}\pi(\delta_{-tf})\,dt\,, \qquad \sigma := \mathit{sign}\,\lambda\,. \tag{15}$$

By an application of this to the GNS–representations of regular states, we also obtain an affine bijection between $\mathfrak{S}_r(\mathcal{R}(X,\sigma))$ *and the regular states* $\mathfrak{S}_r(\overline{\Delta(X,\,\sigma)})$ *of* $\overline{\Delta(X,\,\sigma)}$*, and it restricts to a bijection between the pure regular states of* $\mathcal{R}(X,\sigma)$ *and the pure regular states of* $\overline{\Delta(X,\,\sigma)}$*.*

Note that whilst we have a bijection between the regular states of $\mathcal{R}(X,\sigma)$ and those of $\overline{\Delta(X,\,\sigma)}$, there is no such map between the nonregular states of the two algebras. In fact, fix a nonzero $f \in X$ and consider the two commutative subalgebras $C^*\{R(\lambda,f)\,,\mathbb{1} \mid \lambda \in \mathbb{R}\backslash 0\} \subset \mathcal{R}(X,\sigma)$ and $C^*\{\delta_{tf} \mid t \in \mathbb{R}\} \subset \overline{\Delta(X,\,\sigma)}$, then these are isomorphic respectively to the continuous functions on the one point compactification of $\mathbb{R}$, and the continuous functions on the Bohr compactification of $\mathbb{R}$. Note that the point measures on the compactifications without $\mathbb{R}$ produce nonregular states (after extending to the full C*–algebras by Hahn–Banach) and there are many more of these for the Bohr compactification than for the one point compactification of $\mathbb{R}$, (cf. Theorem 5 in [8, p 949]). So the Weyl algebra has many more nonregular states than the resolvent algebra.

Some further properties of regular representations and states are:

4.5 Proposition *Let* $\mathcal{R}(X,\,\sigma)$ *be as above.*

(i) If a representation π *of* $\mathcal{R}(X,\sigma)$ *is faithful and factorial, it must be regular.*

(ii) If a representation $\pi : \mathcal{R}(X,\sigma) \to \mathcal{B}(\mathcal{H})$ *is regular then* $\|\pi(R(\lambda,f))\| = \|R(\lambda,f)\| = |\lambda|^{-1}$ *for all* $\lambda \in \mathbb{R}\backslash 0,\ f \in X$*.*

(iii) *A state ω of $\mathcal{R}(X,\sigma)$ is regular iff $\omega(A) = \lim_{\lambda\to\infty} i\lambda\,\omega(R(\lambda,f)A)$ for all $A \in \mathcal{R}(X,\sigma)$ and $f \in X$.*

Thus regular states restrict to regular states on subalgebras. Clearly a cyclic component of a regular representation is again regular, so it makes sense to define

4.6 Definition *The universal regular representation of $\mathcal{R}(X,\sigma)$ is*

$$\pi_r := \bigoplus \{\pi_\omega \mid \omega \in \mathfrak{S}_r(\mathcal{R}(X,\sigma))\} \quad \textit{and the regular seminorm is} \quad \|A\|_r := \|\pi_r(A)\|$$

and we denote $\mathcal{R}_r(X,\sigma) := \pi_r(\mathcal{R}(X,\sigma))$.

Now π_r is a subrepresentation of π_u and all regular representations of $\mathcal{R}(X,\sigma)$ will factor through π_r. We want to prove that π_r is faithful, and hence that $\mathcal{R}_r(X,\sigma) \cong \mathcal{R}(X,\sigma)$. For this, we need to develop some structure theory for general representations. First, some notation: for a subspace $S \subset X$ its symplectic complement will be denoted by $S^\perp := \{\, f \in X \mid \sigma(f,S) = 0\,\}$. By $X = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ we will mean that all S_i are nondegenerate and $S_i \subset S_j^\perp$ if $i \neq j$, and each $f \in X$ has a unique decomposition $f = f_1 + f_2 + \cdots + f_n$ such that $f_i \in S_i$ for all i.

4.7 Proposition *Let $\pi : \mathcal{R}(X,\sigma) \to \mathcal{B}(\mathcal{H})$ be a nondegenerate representation. Then*

(i) *the set $X_R := \{\, f \in X \mid \operatorname{Ker}\pi(R(1,f)) = \{0\}\,\}$ is a linear space. Hence if $f \in X_S := X\backslash X_R$, then $f + g \in X_S$ for all $g \in X_R$.*

(ii) *The set $X_T := \Big\{\, f \in X \mid \operatorname{Ker}\pi(R(1,f)) = \{0\}$ and $\pi(R(1,f))^{-1} \in \mathcal{B}(\mathcal{H}) \Big\} \subset X_R$ is a linear space. Moreover if $f \in X_T$ then $\pi(R(1,g)) = 0$ for all $g \in X$ with $\sigma(f,g) \neq 0$. Thus $\sigma(X_T, X_R) = 0$.*

(iii) *If π is factorial, then $\pi(R(1,f)) = 0$ for all $f \in X_S$, and $\pi(R(1,f)) \in \mathbb{C}\mathbb{1}\backslash 0$ for all $f \in X_T$. Moreover $X_T = X_R \cap X_R^\perp$.*

(iv) *Let X be finite dimensional and let $\{q_1,\ldots,q_n\}$ be a basis for X_T. If π is factorial, we can augment this basis of X_T by $\{p_1,\ldots,p_n\} \subset X_S$ into a symplectic basis of $Q := \operatorname{Span}\{q_1,p_1;\ldots;q_n,p_n\}$, i.e. $\sigma(p_i,q_j) = \delta_{ij}$, $0 = \sigma(q_i,q_j) = \sigma(p_i,p_j)$. Then we have the decomposition*

$$X = Q \oplus (Q^\perp \cap X_R) \oplus (Q^\perp \cap X_R^\perp) \tag{16}$$

into nondegenerate spaces such that $Q^\perp \cap X_R \subset \{0\} \cup (X_R\backslash X_T)$ and $Q^\perp \cap X_R^\perp \subset \{0\} \cup X_S$.

Clearly X_R is the part of X on which π is regular, X_T is the part on which it is "trivially regular", X_S is the part on which it is singular, and these have a particularly nice form when π is factorial. So by this Proposition we obtain an interesting regularity structure theory for representations of $\mathcal{R}(X,\sigma)$ which we will exploit below to prove our uniqueness theorem. Note that (ii) implies that if π is regular for a pair f, $g \in X$ with $\sigma(f,g) \neq 0$, then the field operators $\phi_\pi(f)$ and $\phi_\pi(g)$ are both unbounded.

4.8 Proposition *Let X be finite dimensional, and let ω be a nonregular pure state of $\mathcal{R}(X,\sigma)$. Then*

(i) *there is a sequence* $\{\omega_n\} \subset \mathfrak{S}_r(\mathcal{R}(X,\sigma))$ *of regular states such that* $\omega = \lim_{n\to\infty} \omega_n$ *in the w*–topology.*

(ii) $\mathrm{Ker}\, \pi_r \subseteq \mathrm{Ker}\, \pi_\omega$ *where* π_r *is the universal regular representation of* $\mathcal{R}(X,\sigma)$*.*

(iii) $\mathcal{R}(X,\sigma) \cong \mathcal{R}_r(X,\sigma)$*.*

Using this, we can now prove our desired uniqueness theorem, *i.e.* that $\mathcal{R}(X,\sigma) \cong \mathcal{R}_r(X,\sigma)$ for any X.

4.9 Theorem *Let* X *be a nondegenerate symplectic space of arbitrary dimension. Then*

(i) *The norms of* $\mathcal{R}(X,\,\sigma)$ *and* $\mathcal{R}(S,\,\sigma)$ *coincide on *-alg*$\{R(\lambda, f) \mid f \in S,\, \lambda \in \mathbb{R}\backslash 0\}$ *for each finite dimensional nondegenerate subspace* $S \subset X$*. Thus we obtain a containment* $\mathcal{R}(S,\,\sigma) \subset \mathcal{R}(X,\,\sigma)$*.*

(ii) $\mathcal{R}(X,\,\sigma)$ *is the inductive limit of the net of all* $\mathcal{R}(S,\,\sigma)$ *where* $S \subset X$ *ranges over all finite dimensional nondegenerate subspaces of* X*.*

(iii) *We have that* $\mathcal{R}(X,\sigma) \cong \mathcal{R}_r(X,\sigma)$*.*

It follows therefore from Fell's theorem (cf. Theorem 1.2 in [11]) that *any* state of $\mathcal{R}(X,\sigma)$ is in the w*-closure of the convex hull of the vector states of π_r, hence of the regular states. We can now prove the following result, which is relevant for physics.

4.10 Theorem *Let* $(X,\,\sigma)$ *be any nondegenerate symplectic space, and* $\mathcal{R}(X,\sigma)$ *as above. Then every regular representation of* $\mathcal{R}(X,\sigma)$ *is faithful.*

The importance of this result lies in the fact that the regular representations are taken to be the physically relevant ones, and the images of $\mathcal{R}(X,\sigma)$ in all regular representations are isomorphic. Thus, since we can obtain the quantum fields from $\mathcal{R}(X,\sigma)$ in these representations, we are justified in taking $\mathcal{R}(X,\sigma)$ to be the observable algebra for bosonic fields. Usually one argues that for a C*–algebra $\mathcal{A}$ to be an observable algebra of a physical system, it must be simple (cf. [18, p 852]). The argument is that by Fell equivalence of the physical representations, the image of $\mathcal{A}$ in all physical representations must be isomorphic. However, if one restricts the class of physical representations (as we do here to the regular representations of $\mathcal{R}(X,\sigma)$), then the latter isomorphism does not imply that $\mathcal{A}$ must be simple.

This theorem also has structural consequences, *e.g.* it implies that $\mathcal{R}(X,\sigma)$ has faithful irreducible representations, hence that its centre must be trivial. Moreover, it also proves isomorphism with the previous version of the resolvent algebra which we developed in [5], because it was the image of $\mathcal{R}(X,\sigma)$ in the "universal strongly regular representation" which we now define.

For many applications one needs regular representations where there is a dense invariant joint domain for all the fields $\phi_\pi(f)$, and this leads us to a subclass of the regular representations as follows. We will say that a state ω on the Weyl algebra $\overline{\Delta(X,\,\sigma)}$ is strongly regular if the functions

$$\mathbb{R}^n \ni (\lambda_1, \ldots, \lambda_n) \mapsto \omega(\delta_{\lambda_1 f_1} \cdots \delta_{\lambda_n f_n})$$

are smooth for all $f_1, \ldots, f_n \in X$ and all $n \in \mathbb{N}$. Of special importance is that the GNS-representation of a strongly regular state has a common dense invariant domain for all the generators $\phi_{\pi_\omega}(f)$ of the one parameter groups $\lambda \to \pi_\omega(\delta_{\lambda f})$ (this domain is obtained by applying the polynomial algebra of the Weyl operators $\{\, \pi_\omega(\delta_f) \;\mid\; f \in X \,\}$ to the cyclic GNS-vector). By the bijection of Corollary 4.4, we then obtain the set of strongly regular states on $\mathcal{R}(X, \sigma)$, and we denote this by $\mathfrak{S}_{sr}(\mathcal{R}(X, \sigma))$.

4.11 Definition *The universal strongly regular representation of $\mathcal{R}(X, \sigma)$ is*

$$\pi_{sr} := \bigoplus \{\pi_\omega \;\mid\; \omega \in \mathfrak{S}_{sr}(\mathcal{R}(X, \sigma))\}\,.$$

$\pi_{sr}(\mathcal{R}(X, \sigma))$ is the version of the resolvent algebra which we used in [5], which is obviously isomorphic to $\mathcal{R}(X, \sigma)$ by Theorem 4.10.

An important set of strongly regular states on $\overline{\Delta(X,\, \sigma)}$ are the quasifree states. They are given by

$$\omega(\delta_f) = \exp\big(- \tfrac{1}{2} \langle f | f \rangle_\omega\big), \quad f \in X,$$

where $\langle \cdot | \cdot \rangle_\omega$ is a (possibly semi–definite) scalar product on the complex linear space $X + iX$ satisfying

$$\langle f | g \rangle_\omega - \langle g | f \rangle_\omega = i\sigma(f, g), \quad f, g \in X.$$

By a routine computation one can represent the expectation values of products of Weyl operators in a quasifree state in the form

$$\omega(\delta_{f_1} \cdots \delta_{f_n}) = \exp\Big(- \sum_{k<l} \langle f_k | f_l \rangle_\omega - \tfrac{1}{2} \sum_l \langle f_l | f_l \rangle_\omega \Big).$$

Making use of the Laplace transform (15) for the GNS–representation of the resolvents, we have for $\lambda_1, \ldots \lambda_n > 0$

$$\begin{aligned}
&\omega(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n)) \\
&= (-i)^n \int_0^\infty dt_1 \ldots \int_0^\infty dt_n\, e^{-\sum_k t_k \lambda_k} \omega(\delta_{t_1 f_1} \cdots \delta_{t_n f_n}) \\
&= (-i)^n \int_0^\infty dt_1 \ldots \int_0^\infty dt_n \, \exp\Big(- \sum_k t_k \lambda_k - \sum_{k<l} t_k t_l \langle f_k | f_l \rangle_\omega - \tfrac{1}{2} \sum_l t_l^2 \langle f_l | f_l \rangle_\omega \Big). \qquad (17)
\end{aligned}$$

4.12 Remark One can replace anywhere in this equation f_k by $-f_k$, thus it does not impose any restriction of generality to assume that $\lambda_1, \ldots \lambda_n > 0$. The relation (17) should be regarded as the definition of quasifree states on the resolvent algebra.

Concerning unitary implementability of automorphisms, we have the following easy fact:

4.13 Proposition *Let $\alpha \in \operatorname{Aut} \mathcal{R}(X, \sigma)$ correspond to a symplectic transformation $T \in \operatorname{Sp}(X, \sigma)$ by $\alpha(R(\lambda, f)) := R(\lambda, Tf)$ for $f \in X$. Then α is implemented by a unitary in both π_r and π_{sr}.*

5 Further structure.

Here we want to explore the algebraic structure of $\mathcal{R}(X,\,\sigma)$.

5.1 Theorem *Let $(X,\,\sigma)$ be a given nondegenerate symplectic space, and let $X = S \oplus S^\perp$ for $S \subset X$ a nondegenerate subspace. Then*

$$\mathcal{R}(X,\,\sigma) \supset \mathrm{C}^*(\mathcal{R}(S,\,\sigma) \cup \mathcal{R}(S^\perp,\,\sigma)) \cong \mathcal{R}(S,\,\sigma) \otimes \mathcal{R}(S^\perp,\,\sigma)$$

where the tensor product uses the minimal (spatial) tensor norm. The containment is proper in general.

Thus we cannot generate $\mathcal{R}(X,\sigma)$ from a basis alone, *i.e.* if $\{q_1, p_1;\, q_2, p_2; \cdots\}$ is a symplectic basis of X, then $\mathrm{C}^*\{R(\lambda_i, q_i),\ R(\mu_i, p_i) \mid \lambda_i,\, \mu_i \in \mathbb{R}\backslash 0,\ i = 1,\, 2, \ldots\}$ is in general a proper subalgebra of $\mathcal{R}(X,\sigma)$, though in any regular representation π it is strong operator dense in $\pi(\mathcal{R}(X,\sigma))$ by Theorem 4.2(v).

Note that since $C^*(\{R(\lambda, f) \mid \lambda \in \mathbb{R}\backslash 0\}) \cong C_0(\mathbb{R})$ (easily seen in any regular representation), and we have that $C_0(\mathbb{R}^{n+m}) = C_0(\mathbb{R}^n) \otimes C_0(\mathbb{R}^m)$, it follows from Theorem 5.1 that any C_0–function of a finite commuting set of variables is in $\mathcal{R}(X,\sigma)$. More concretely, we have the following result which will be used later.

5.2 Proposition *Let $\{q_1, \ldots,\, q_k\} \subset X$ satisfy $\sigma(q_i, q_j) = 0$ for all $i,\, j$. Then for each $F \in C_0(\mathbb{R}^k)$ there is a (unique) $R_F \in \mathcal{R}(X,\sigma)$ such that in any regular representation π we have $\pi(R_F) = F(\phi_\pi(q_1), \ldots, \phi_\pi(q_k))$.*

Thus the resolvent algebra contains in abstract form all C_0–functions of commuting fields. Note that such a result neither holds for the Weyl algebra nor for the corresponding twisted group algebra (in the case of finite dimensional X).

5.3 Theorem *Let $(X,\,\sigma)$ be a given nondegenerate symplectic space and let $f,\, h \in X\backslash 0$ such that $f \notin \mathbb{R}h$. Then*

(i) $R(1, f) \notin [\mathcal{R}(X,\sigma)R(1, h)]$, i.e. the ideals separate the rays of X,

(ii) $\|R(1, f) - R(1, h)\| \geq 1$, and if $\sigma(f, h) = 0$ we have equality.

(iii) $\mathcal{R}(X,\sigma)$ is nonseparable.

Next, let us assume that our symplectic space X is finite dimensional, hence by the von Neumann uniqueness theorem there is (up to unitary equivalence) a unique irreducible regular representation π_0 of $\mathcal{R}(X,\sigma)$.

5.4 Theorem *Let $(X,\,\sigma)$ be a given finite dimensional nondegenerate symplectic space equipped with the symplectic basis $\{q_1, p_1; \ldots; q_n, p_n\}$, and let $\pi_0 : \mathcal{R}(X,\sigma) \to \mathcal{B}(\mathcal{H}_0)$ be an irreducible regular representation of $\mathcal{R}(X,\sigma)$. Then*

(i) $\pi_0\Big((R(\lambda_1, p_1)R(\mu_1, q_1)) \cdots (R(\lambda_n, p_n)R(\mu_n, q_n))\Big)$ is a Hilbert–Schmidt operator for all λ_i, $\mu_i \in \mathbb{R}\backslash 0$.

(ii) *There is a unique closed two-sided ideal* $\mathcal{K}$ *of* $\mathcal{R}(X,\sigma)$ *which is isomorphic to the algebra of compact operators* $\mathcal{K}(\mathcal{H}_0)$*, and such that* $\pi_0(\mathcal{K}) = \mathcal{K}(\mathcal{H}_0) \subset \mathcal{B}(\mathcal{H}_0)$*.*

(iii) *A representation of* $\mathcal{R}(X,\sigma)$ *is regular iff its restriction to* $\mathcal{K}$ *is nondegenerate. Thus the regular representations (resp. regular states) are exactly the unique extensions of the representations (resp. states) of* $\mathcal{K}$ *to* $\mathcal{R}(X,\sigma)$*.*

(iv) *If* $n = 1$ *then the factor algebra* $\mathcal{R}(X,\sigma)/\mathcal{K}$ *is commutative, but not if* $n > 1$*.*

(v) $\mathcal{K}$ *is an essential ideal, i.e. if* $A\mathcal{K} = 0$ *or* $\mathcal{K}A = 0$*, then* $A = 0$*.*

(vi) $\mathcal{K}$ *is a minimal (nonzero) closed two–sided ideal, and is contained in every closed nonzero two–sided ideal of* $\mathcal{R}(X,\sigma)$*. Thus all closed nonzero two–sided ideals of* $\mathcal{R}(X,\sigma)$ *are essential.*

5.5 Remarks (a) The statement in (iii) that the regular representation theory of $\mathcal{R}(X,\sigma)$ is the representation theory of the compacts, is of course just a paraphrasing of the von Neumann uniqueness theorem. In fact, it is well–known that the C*–closure of the twisted convolution algebra $L^1(X)$ w.r.t. $\exp(i\sigma)$ (which is the twisted group algebra of X thought of as an abelian group) is isomorphic to the compacts, and that the Weyl algebra acts on it by multipliers (cf. first part of [14] for a discussion of this). The regular representations on the Weyl algebra are likewise obtained for finite dimensional X from the unique extensions of the representations of the compacts. The main attractive feature of the resolvent algebra is that the algebra of compacts is actually an ideal inside $\mathcal{R}(X,\sigma)$, in contrast with the Weyl algebra where it is outside. This is very useful, and we will utilize this fact below.

(b) From the structure above, we note that the nonregular representations are those which have a direct summand which is the restriction of a representation of the Calkin algebra to $\mathcal{R}(X,\sigma)$.

(c) From (vi) and (iii) we obtain a quick proof of Theorem 4.10 as follows. Let π be a regular representation of $\mathcal{R}(X,\sigma)$ where X is arbitrary. Since $\mathcal{R}(X,\sigma)$ is an inductive limit of $\mathcal{R}(S,\sigma)$ where $S \subset X$ ranges over the finite dimensional nondegenerate subspaces, it suffices to show that π is faithful on each $\mathcal{R}(S,\sigma)$. But if π is not faithful on a $\mathcal{R}(S,\sigma)$ with $\dim(S) < \infty$, then by (vi) π must vanish on $\mathcal{K}$ hence cannot be regular by (iii) which is a contradiction.

(d) If $n > 1$ then $\mathcal{R}(X,\sigma)$ contains infinitely many copies of the compacts, because for each nondegenerate proper subspace $S \subset X$, the copy of $\mathcal{R}(S,\sigma)$ in $\mathcal{R}(X,\sigma)$ will contain its own compact ideal $\mathcal{K}_S \subset \mathcal{R}(S,\sigma)$. In general these will not be ideals of $\mathcal{R}(X,\sigma)$, nor will they map onto the compacts in the irreducible regular representation. This can be seen from the fact that $X = S \oplus S^\perp$, hence $\mathcal{R}(S,\sigma)$ is embedded as $\mathcal{R}(S,\sigma) \otimes \mathbb{1}$ in the subalgebra $\mathcal{R}(S,\sigma) \otimes \mathcal{R}(S^\perp,\sigma)$ of $\mathcal{R}(X,\sigma)$. It is now clear that $\mathcal{K}_S \otimes \mathbb{1}$ is not an ideal, and in the Schrödinger representation w.r.t. the union of a symplectic basis of S and of $S^\perp$ that $\mathcal{K}_S \otimes \mathbb{1}$ maps to a tensor product of compacts with the identity, which is not compact. These embedded copies of $\mathcal{K}$ are nevertheless useful as indicators of partial regularity, *i.e.* a representation π is regular on a nondegenerate subspace $S \subset X$ iff it is nondegenerate on $\mathcal{K}_S$. This can be particularly useful for the infinite dimensional

case, where we do not have a $\mathcal{K} \subset \mathcal{R}(X,\sigma)$, but we still have that π is regular iff its restriction to $\mathcal{K}_S$ is nondegenerate for each finite dimensional nondegenerate subspace $S \subset X$.

If X is infinite dimensional, the question naturally arises as to whether there is a C*–algebra $\mathcal{L}$ which can play the role of $\mathcal{K}$. That is, we want at least that there is a faithful embedding of $\mathcal{R}(X,\sigma)$ in the multiplier algebra $M(\mathcal{L})$ and such that the unique extensions of representations from $\mathcal{L}$ to $\mathcal{R}(X,\sigma) \subset M(\mathcal{L})$ produces an identification of the representation theory of $\mathcal{L}$ with the regular representations of $\mathcal{R}(X,\sigma)$. In the case that S is countably dimensional such an algebra has recently been constructed for the Weyl algebra, cf. [17], hence is a strong candidate.

6 Dynamics and Hamiltonians

For the dynamics (*i.e.* time evolution) of a quantum mechanical system, one usually assumes a one–parameter automorphism group of the observable algebra, and considers distinguished representations in which it is implemented by strong operator continuous unitary groups. Much analysis is then done of the generators (Hamiltonians). On both the Weyl algebra $\overline{\Delta(X,\,\sigma)}$ and the resolvent algebra $\mathcal{R}(X,\sigma)$ one can define dynamical groups in terms of symplectic transformations. However, since such dynamics correspond to quadratic Hamiltonians, it does not cover most physically interesting situations. In fact, for the Weyl algebra it is a well–known frustration that many natural time–evolutions cannot be defined on it. For example in [10] it is proven for the case $X = \mathbb{R}^2$ that time evolutions obtained from Hamiltonians of the form $H = P^2 + V(Q)$ for potentials $V \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ do not preserve $\overline{\Delta(X,\,\sigma)}$ in the Schrödinger representation unless $V = 0$. Moreover, there is no adequate method known by which one can somehow encode the dynamics (or even more desirably, the Hamiltonian) in the Weyl algebra.

We want to show here that for the resolvent algebra $\mathcal{R}(X,\sigma)$ the situation is completely different. We will not give a comprehensive analysis of the issue of dynamics but will rather illustrate the technical benefits of the resolvent algebra for the analysis of dynamics questions. To simplify matters, we will assume in the rest of this section that $X = \mathbb{R}^2$ with its standard symplectic form σ and work in the Schrödinger representation of $\mathcal{R}(X,\sigma)$. With some effort, our results can be extended to the case of arbitrary finite dimensional and non–degenerate symplectic spaces X. In our example in the next section we will touch on dynamics (with interaction) when X is infinite dimensional.

6.1 Dynamics on $\mathcal{R}(X,\sigma)$

Henceforth, for this section, let $X = \mathbb{R}^2$ with its standard symplectic form. Since the Schrödinger representation π_0 is faithful on $\mathcal{R}(X,\sigma)$, the dynamics on $\mathcal{R}(X,\sigma)$ can be defined in π_0 as follows. Let H be a selfadjoint operator (Hamiltonian) generating the unitary group $U(t) = e^{itH}$, $t \in \mathbb{R}$. In favourable cases where

$$U(t)\pi_0(\mathcal{R}(X,\sigma))U(t)^{-1} \subset \pi_0(\mathcal{R}(X,\sigma))\,, \quad t \in \mathbb{R},$$

one can define a corresponding automorphic action α_t on $\mathcal{R}(X, \sigma)$, $t \in \mathbb{R}$ putting

$$\alpha_t(R) := \pi_0^{-1}(U(t)\pi_0(R)U(t)^{-1}), \quad R \in \mathcal{R}(X, \sigma).$$

We will say in these cases that the Hamiltonian H induces a dynamics on the resolvent algebra $\mathcal{R}(X, \sigma)$ (where the context of π_0 is assumed). From the case when H is quadratic, we see via Theorem 5.3(ii) that in general the actions $t \mapsto \alpha_t$ need not be pointwise norm–continuous.

The Hamiltonians which induce a dynamics on $\mathcal{R}(X, \sigma)$ exist in abundance. Denote by Q, P the canonical position and momentum operators in the Schrödinger representation, then we have the following result in contrast with the no–go theorem for the Weyl algebra above.

6.1 Proposition *Let $V \in C_0(\mathbb{R})$ be any real function. Then the corresponding selfadjoint Hamiltonian $H = P^2 + V(Q)$ induces a dynamics on $\mathcal{R}(X, \sigma)$.*

Thus the resolvent algebra is an appropriate framework for the formulation of dynamical laws.

6.2 Hamiltonians affiliated with $\mathcal{R}(X, \sigma)$

It is another interesting feature of the resolvent algebra $\mathcal{R}(X, \sigma)$ that it contains many observables of physical interest. We illustrate this fact within the preceding concrete setting.

Let H be a selfadjoint operator. When its resolvent is contained in $\pi_0(\mathcal{R}(X, \sigma))$ we may proceed to its pre–image,

$$R_\lambda := \pi_0^{-1}((i\lambda \mathbb{1} - H)^{-1}), \quad \lambda \in \mathbb{R}\backslash\{0\},$$

defining a pseudo–resolvent in $\mathcal{R}(X, \sigma)$. By a slight abuse of terminology we say then that H is affiliated with $\mathcal{R}(X, \sigma)$ and regard it as an observable which is predetermined by the resolvent algebra. In the regular representation π_0 the pseudo–resolvent R_λ produces the selfadjoint Hamiltonian H which is the energy operator of the underlying states. But R_λ can also be used for the determination of the energy content of non–regular representations including extreme cases where the energy of all states is infinite and R_λ is represented by 0. The next result exhibits a multitude of Hamiltonians which are affiliated with the resolvent algebra.

6.2 Proposition *Let $V \in C_0(\mathbb{R})$ be any real function. Then the corresponding selfadjoint Hamiltonian $H = P^2 + V(Q)$ is affiliated with $\mathcal{R}(X, \sigma)$.*

Since X is finite dimensional $\mathcal{R}(X, \sigma)$ also contains the compact operators and hence in particular all one–dimensional projections. As these projections play a fundamental role in quantum systems, we conclude that the resolvent algebra contains all necessary ingredients for the treatment of quantum mechanical systems. It therefore qualifies as a genuine observable algebra.

6.3 Hamiltonians not affiliated with $\mathcal{R}(X, \sigma)$

Since $\pi_0(\mathcal{R}(X, \sigma))$ is a proper subalgebra of the algebra of all bounded operators on the underlying representation space it is clear from the outset that there are many selfadjoint operators which

are not affiliated with $\mathcal{R}(X, \sigma)$. It is thus of interest to explore which operators do not qualify as observables in the framework of the resolvent algebra. We do not have a complete answer to this question, not even for the case of quantum mechanical Hamiltonians. But the next result shows that many Hamiltonians which seem physically unacceptable can be excluded this way.

6.3 Proposition *The selfadjoint Hamiltonian $H = P^2 - Q^2$ is not affiliated with $\mathcal{R}(X, \sigma)$.*

Since this Hamiltonian is not semibounded it does not describe a stable system and hence is of limited physical interest. Note that, in contrast, the Hamiltonian $H = P^2 + Q^2$ of the harmonic oscillator is an observable which is affiliated with $\mathcal{R}(X, \sigma)$ since its resolvent is a compact operator.

7 Infinite dimensional dynamical systems

The resolvent algebra exhibits its full power when dealing with systems for which $\dim(X)$ is infinite. For, in contrast to the finite dimensional case, there exists then an abundance of disjoint regular representations. It is a notorious problem in quantum physics to select the representations which describe the states of interest for a given infinite dimensional system. This selection is usually done by the specification of a dynamics, and searching for representations describing specific situations such as ground states or thermal equilibrium states. It is an asset of the resolvent algebra that it admits the definition of interesting dynamics also in the case of infinite dimensional systems. Moreover, it simplifies the construction of states and representations of interest by the use of C*–algebra techniques.

Next, we develop a concrete model to illustrate these facts. Our model describes particles which are confined around the points of a lattice by harmonic pinning forces and which interact with their nearest neighbours. Thus it resembles the situation in quantum spin systems [4], but in contrast to the latter well–understood class of theories, the present model has an infinite number of degrees of freedom at each lattice site.

Turning to mathematics, let (X, σ) be a countably dimensional symplectic space with a fixed symplectic basis $\{p_l, q_l \mid l \in \mathbb{Z}\}$, where the index $l \in \mathbb{Z}$ labels the lattice points. Denote $X_l := \mathrm{Span}\{p_l, q_l\}$ and $X_\Lambda := \mathrm{Span}\{p_l, q_l \mid l \in \Lambda\}$, where $\Lambda \subset \mathbb{Z}$ is any finite subset of lattice points. Then we take the observable algebra of the model to be $\mathcal{R}(X, \sigma)$, and it is the C*–inductive limit of the algebras $\mathcal{R}(X_\Lambda, \sigma)$, $\Lambda \subset \mathbb{Z}$.

To define the dynamics on $\mathcal{R}(X, \sigma)$ we fix a regular (hence faithful) representation π_0 of $\mathcal{R}(X, \sigma)$, and here we will take π_0 to be the Fock representation w.r.t. the given basis. Recall that this Fock representation is the (irreducible) representation which is determined by the fact that there exits a unit vector $\Omega_0 \in \mathcal{H}_0$ in the domain of all polynomials of the fields satisfying

$$(\phi_{\pi_0}(p_l) + i\phi_{\pi_0}(q_l))\,\Omega_0 = 0\,, \quad l \in \mathbb{Z}\,.$$

As is well–known, this vector defines a product state on the algebra $\overline{\Delta(X,\, \sigma)}$ in the sense that if S and T are nondegenerate subspaces of X with $S \cap T = \{0\}$ then

$$(\Omega_0, \pi_0(W_1)\pi_0(W_2)\Omega_0) = (\Omega_0, \pi_0(W_1)\Omega_0)\,(\Omega_0, \pi_0(W_2)\Omega_0)\,, \quad W_1 \in \overline{\Delta(S,\, \sigma)},\ W_2 \in \overline{\Delta(T,\, \sigma)}\,.$$

By strong operator limits and continuity we obtain the analogous statement for $\mathcal{R}(X, \sigma)$ *i.e.* Ω_0 also defines a product state for $\mathcal{R}(X, \sigma)$ in a similar sense.

As a consequence, for any nondegenerate subspace $S \subset X$ the restriction $\pi_0 \upharpoonright \mathcal{R}(S, \sigma)$ acts irreducibly on the corresponding subspace $\mathcal{H}_0(S) = \overline{\pi_0(\mathcal{R}(S, \sigma))\,\Omega_0}$ and it is equivalent to the Fock representation of $\mathcal{R}(S, \sigma)$. Moreover $\pi_0(\mathcal{R}(S, \sigma))''$ is a factor. In particular, these statements hold for $S = X_\Lambda$, $\Lambda \subset \mathbb{Z}$. Even though our results do not depend on the choice of representation π_0, its specific features will greatly simplify the necessary computations.

7.1 Interacting dynamical systems

Turning to the problem of defining the dynamics, denote:

$$Q_l := \phi_{\pi_0}(p_l), \ P_l := \phi_{\pi_0}(q_l), \quad l \in \mathbb{Z}.$$

Let $V \in C_0(\mathbb{R})$ be any real function which models the interaction potential. We consider for each $\Lambda \subset \mathbb{Z}$ the selfadjoint "local Hamiltonian"

$$H_\Lambda := \sum_{l \in \Lambda} (P_l^2 + Q_l^2) + \sum_{l,\, l+1 \in \Lambda} V(Q_l - Q_{l+1}),$$

describing the interaction amongst the particles in Λ. The dynamics on the full system is then obtained as follows.

7.1 Proposition *Let $V \in C_0(\mathbb{R})$ be a real function and let $\{H_\Lambda\}_{\Lambda \subset \mathbb{Z}}$ be the corresponding set of Hamiltonians as above, and define the unitary groups $U_\Lambda(t) := e^{itH_\Lambda}$, $t \in \mathbb{R}$. Then*

(i) $U_\Lambda(t)\, \pi_0(\mathcal{R}(X_\Lambda, \sigma))\, U_\Lambda(t)^{-1} \subset \pi_0(\mathcal{R}(X_\Lambda, \sigma))$ *for all* $t \in \mathbb{R}$ *and* $\Lambda \subset \mathbb{Z}$.

(ii) For any $R \in \mathcal{R}(X_{\Lambda_0}, \sigma)$, $\Lambda_0 \subset \mathbb{Z}$ *the net* $\{U_\Lambda(t)\pi_0(R)U_\Lambda(t)^{-1}\}_{\Lambda \subset \mathbb{Z}}$ *converges in norm to an element of* $\pi_0(\mathcal{R}(X, \sigma))$ *as* $\Lambda \nearrow \mathbb{Z}$.

(iii) There is a unique automorphic action (dynamics) $\alpha : \mathbb{R} \to \mathrm{Aut}\,(\mathcal{R}(X, \sigma))$ *such that for any* $\Lambda_0 \subset \mathbb{Z}$ *and* $R \in \mathcal{R}(X_{\Lambda_0}, \sigma)$ *we have*

$$\alpha_t(R) := n\text{–}\lim_{\Lambda \nearrow \mathbb{Z}} \pi_0^{-1}(U_\Lambda(t)\pi_0(R)U_\Lambda(t)^{-1})$$

where n–lim *denotes the norm limit.*

This result can be extended to lattice models in higher dimensions. Moreover, the harmonic pinning potentials Q^2 may be dropped from the local Hamiltonians without changing the result. For the modelling of interacting systems of indistinguishable Bosons one would have to replace the nearest neighbour interaction by a full two–body potential. Whereas the stability of the C*–algebras $\pi_0(\mathcal{R}(X_\Lambda, \sigma))$ under the action of the corresponding local dynamics can still be established in that case by the present methods, the existence of the thermodynamic limit $\Lambda \nearrow \mathbb{Z}$ is not settled. We hope to return to this problem elsewhere.

We would like to emphasize in conclusion that the action $\alpha : \mathbb{R} \to \mathrm{Aut}\,(\mathcal{R}(X, \sigma))$ is *not* pointwise norm–continuous, *i.e.* we are not in the usual setting of C*–dynamical systems. However

there is an important substitute: Let $\mathcal{K}_\Lambda$ be the compact ideal in $\mathcal{R}(X_\Lambda, \sigma)$, $\Lambda \subset \mathbb{Z}$ and consider the C*–algebra $\mathcal{K} = C^*\{\mathcal{K}_\Lambda \,|\, \Lambda \subset \mathbb{Z}\} + \mathbb{C}\mathbb{1} \subset \mathcal{R}(X, \sigma)$. This algebra is a proper subalgebra of $\mathcal{R}(X, \sigma)$, but it is weakly dense in $\mathcal{R}(X, \sigma)$ in any regular representation. (This fact together with the next result will be useful for the analysis of states in the next subsection.) Then:

7.2 Proposition *Let $V \in C_0(\mathbb{R})$ be a real function and let $\{H_\Lambda\}_{\Lambda\subset\mathbb{Z}}$ be the corresponding family of Hamiltonians. Then the action $\alpha : \mathbb{R} \to \mathrm{Aut}\,(\mathcal{R}(X, \sigma))$ defined above is pointwise norm–continuous on the elements of $\mathcal{K} \subset \mathcal{R}(X, \sigma)$. (Note that $\mathcal{K}$ is in general not stable under the action of $\alpha_\mathbb{R}$.)*

It is an immediate consequence of this proposition that the C*–algebra $\mathcal{L}$ generated by the algebras $\alpha_t(\mathcal{K})$, $t \in \mathbb{R}$ is stable under the action of of $\alpha_\mathbb{R}$, and $\alpha : \mathbb{R} \to \mathrm{Aut}\,(\mathcal{R}(X, \sigma))$ acts pointwise norm continuously on $\mathcal{L}$. Thus $\mathcal{L}$ depends on the chosen dynamics and will be different, for example, for lattice systems with a next to nearest neighbour interaction. It seems that there is no universal subalgebra of the resolvent algebra which is stable and pointwise norm continuous under the action of all possible dynamics.

7.2 Ground states of interacting dynamical systems

Having established the existence of a large family of interacting dynamics $\alpha_\mathbb{R}$ on $\mathcal{R}(X, \sigma)$ we want to analyze now whether there exist corresponding states of physical interest. We will focus on ground states; but our arguments also apply to thermal equilibrium states. It is clear from the outset by Haag's theorem [9] that we will have to leave the Fock representation π_0 in order to describe the desired states. As we will see, it is very simple to identify the correct representations and establish their desired properties in the present framework.

Fix the potential $V \in C_0(\mathbb{R})$ and let $\{H_\Lambda\}_{\Lambda\subset\mathbb{Z}}$ be the corresponding set of local Hamiltonians. The operators H_Λ differ from the Hamiltonians $H_\Lambda^{(0)}$ of the harmonic oscillator for Λ by the bounded interaction potential which is an element of $\mathcal{R}(X_\Lambda, \sigma)$. Thus, since the restriction of the resolvent of $H_\Lambda^{(0)}$ to $\mathcal{H}_0(X_\Lambda)$ is a compact operator, so is the restriction of the resolvent of H_Λ. Hence, by the same reasoning as in Sec. 6.2, it follows that H_Λ is an observable which is affiliated with the compact ideal $\mathcal{K}_\Lambda \subset \mathcal{R}(X_\Lambda, \sigma)$. In particular, each H_Λ has discrete spectrum and, moreover, is bounded from below. It suffices to consider here the Hamiltonians $H_n := H_{\Lambda_n}$ corresponding to the sets $\Lambda_n := \{l \in \mathbb{Z} \mid |l| \le n\}$, $n \in \mathbb{N}$. Let E_n be the smallest eigenvalue of H_n and let $\widetilde{H}_n := H_n - E_n\mathbb{1}$ be the corresponding "renormalized" Hamiltonian whose smallest eigenvalue is 0.

Fix for each $n \in \mathbb{N}$ a normalized eigenvector Ω_n corresponding to the eigenvalue 0 of $\widetilde{H}_n$, and define a sequence of states $\{\omega_n\}_{n\in\mathbb{N}}$ on $\mathcal{R}(X, \sigma)$ by

$$\omega_n(R) := (\Omega_n, \pi_0(R)\Omega_n)\,, \quad R \in \mathcal{R}(X, \sigma)\,.$$

This sequence need not converge, but by w*–compactness of the unit ball of the dual of $\mathcal{R}(X, \sigma)$ it has limit points. Let ω_∞ be any such w*–limit point. Since ω_∞ is a limit of ground states of the local Hamiltonians it is a candidate for a ground state of the full theory; but for infinite

systems such heuristic expectations are known to be taken with a pinch of salt and to require careful analysis. Our first result shows that ω_∞ is a physically acceptable state. It is essential for its proof that the local Hamiltonians are affiliated with the resolvent algebra.

7.3 Lemma *Let ω_∞ be any w*–limit point of the sequence $\{\omega_n\}_{n\in\mathbb{N}}$ defined above. Then ω_∞ is a regular state on $\mathcal{R}(X,\sigma)$.*

With this information we can prove that ω_∞ is a ground state for the dynamics $\alpha : \mathbb{R} \to \mathrm{Aut}\,(\mathcal{R}(X,\sigma))$. We first show that ω_∞ is invariant under the adjoint action of α. Fix $R \in \mathcal{R}(X,\sigma)$ and $t \in \mathbb{R}$, then by definition of ω_∞ there is a subsequence $\{\omega_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\omega_\infty(\alpha_t(R) - R) = \lim_{k\to\infty} \omega_{n_k}(\alpha_t(R) - R) = \lim_{k\to\infty} (\Omega_{n_k}, \pi_0(\alpha_t(R) - R)\Omega_{n_k}) .$$

By the results of the preceding section we have the norm limits

$$\pi_0(\alpha_t(R)) = \text{n–}\lim_{k\to\infty} (\mathrm{Ad}\ e^{itH_{n_k}})(\pi_0(R)) = \text{n–}\lim_{k\to\infty} (\mathrm{Ad}\ e^{it\widetilde{H}_{n_k}})(\pi_0(R)) ,$$

where the second equality follows from the fact that the additive renormalization term in $\widetilde{H}_{n_k}$ drops out in the adjoint action. But $(\Omega_{n_k}, (\mathrm{Ad}\ e^{it\widetilde{H}_{n_k}})(\pi_0(R))\Omega_{n_k}) = (\Omega_{n_k}, \pi_0(R)\Omega_{n_k})$ since Ω_{n_k} is a ground state vector for $\widetilde{H}_{n_k}$. Hence $\omega_\infty(\alpha_t(R) - R) = 0$ for $R \in \mathcal{R}(X,\sigma)$, $t \in \mathbb{R}$.

In the GNS–representation $(\pi_\infty, \mathcal{H}_\infty, \Omega_\infty)$ of the α–invariant state ω_∞, we can now define as usual a unitary representation U_∞ of $\mathbb{R}$ implementing the action α by

$$U_\infty(t)\pi_\infty(R)\Omega_\infty := \pi_\infty(\alpha_t(R))\Omega_\infty , \quad R \in \mathcal{R}(X,\sigma),\, t \in \mathbb{R} .$$

It follows from Proposition 7.2 that $U_\infty(\mathbb{R})$ acts continuously on the subspace $\pi_\infty(\mathcal{K})\Omega_\infty \subset \mathcal{H}_\infty$. But ω_∞ is a regular state, so this subspace is dense in $\mathcal{H}_\infty$, proving that the representation U_∞ is continuous in the strong operator topology.

It remains to determine the spectrum of $U_\infty(\mathbb{R})$. Let $h \in \mathcal{S}(\mathbb{R})$ be a test function whose Fourier transform has support on the negative real axis and let $K \in \mathcal{K}$. By Proposition 7.2 the integral $K(h) := \int dt\, h(t)\alpha_t(K)$ is defined in the norm topology and hence an element of $\mathcal{R}(X,\sigma)$. Picking any other $R \in \mathcal{R}(X,\sigma)$ there is a subsequence $\{\omega_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\omega_\infty(RK(h)) = \lim_{k\to\infty} \omega_{n_k}(RK(h)) = \lim_{k\to\infty} (\Omega_{n_k}, \pi_0(RK(h))\Omega_{n_k}) .$$

Furthermore, making use once more of the results in the preceding section and the dominated convergence theorem we have

$$\pi_0(K(h)) = \int dt\, h(t)\pi_0(\alpha_t(K)) = \text{n–}\lim_{k\to\infty} \int dt\, h(t)(\mathrm{Ad}\ e^{it\widetilde{H}_{n_k}})(\pi_0(K)) .$$

Combining these facts and $e^{-it\widetilde{H}_{n_k}}\,\Omega_{n_k} = \Omega_{n_k}$ we get

$$\omega_\infty(RK(h)) = \lim_{k\to\infty} \int dt\, h(t)\, (\Omega_{n_k}, \pi_0(R)e^{it\widetilde{H}_{n_k}}\pi_0(K)\Omega_{n_k}) = 0 ,$$

where for the second equality we made use of the support properties of the Fourier transform of h and the fact that each $\widetilde{H}_{n_k}$ is a positive operator. Since $\omega_\infty(RK(h)) = \int dt\, h(t)\omega_\infty(R\alpha_t(K))$

by Proposition 7.2 it follows that $\int dt\, h(t)\, U_\infty(t) \restriction \pi_0(\mathcal{K})\Omega_\infty = 0$. But $\pi_0(\mathcal{K})\Omega_\infty$ is dense in $\mathcal{H}_\infty$, so this equality holds on the whole Hilbert space. We have thus shown that the generator of $U_\infty(\mathbb{R})$ has spectrum on the positive real axis, proving that Ω_∞ is a ground state vector. The preceding results are summarized in the next proposition.

7.4 Proposition *Let ω_∞ be any w*–limit point of the sequence $\{\omega_n\}_{n\in\mathbb{N}}$ constructed above. Then ω_∞ is a regular ground state for the dynamics $\alpha : \mathbb{R} \to \mathrm{Aut}\,(\mathcal{R}(X,\sigma))$.*

Here we end our discussion of infinite dynamical systems in the framework of the resolvent algebra. There are many more intriguing questions which can be addressed in the present model, such as the uniqueness and purity of ω_∞, its behaviour under lattice translations (which are a symmetry of the present model) *etc*. But we think that the results presented so far provide sufficient evidence that the resolvent algebra is a natural and powerful setting for the analysis of infinite dynamical quantum systems.

8 Constraint theory.

In this section we illustrate the usefulness of the resolvent algebra in the study of the non–regular representations which occur naturally in some applications of physical interest. Nonregular representations of the CCR–algebra have been used in a number of situations, cf. [1]. In particular they occur in the context of constraint theory, cf. [15,16], and they have been used to circumvent indefinite metric representations in Gupta–Bleuler electromagnetism, cf. [16]. In this section, we wish to develop constraint theory for linear bosonic constraints to see what form it takes in the resolvent algebra, and to investigate the nonregular representations which occur. Since the constraints eliminate nonphysical information, the nonregularity of the representations is not a problem. One only needs regularity on the final physical algebra.

For linear bosonic constraints, we start with a nondegenerate symplectic space (X,σ) and specify a nonzero *constraint subspace* $C \subset X$. Our task is to implement the heuristic constraint conditions

$$\phi(f)\,\Psi = 0 \quad f \in C$$

to select the subspace spanned by the physical vectors Ψ. There are many examples where these occur, *e.g.* in quantum electromagnetism, cf. [12,13,16]. Now in a representation π of $\mathcal{R}(X,\sigma)$ for which $\mathrm{Ker}\,\pi(R(\lambda,f)) = \{0\}$ we have by Theorem 4.2(vi) that $\pi(R(\lambda,f))\phi_\pi(f) = i\lambda\pi(R(\lambda,f)) - \mathbb{1}$ on $\mathrm{Dom}\,\phi_\pi(f)$. Hence the appropriate form in which to impose the heuristic constraint condition in the resolvent algebra is to select the set of physical ("Dirac") states by

$$\mathfrak{S}_D := \{\omega \in \mathfrak{S}(\mathcal{R}(X,\sigma)) \mid \pi_\omega(i\lambda R(\lambda,f) - \mathbb{1})\,\Omega_\omega = 0 \quad f \in C,\ \lambda \in \mathbb{R}\backslash 0\}\ , \tag{18}$$

where π_ω and Ω_ω denote the GNS–representation and GNS–cyclic vector of ω. Thus $\omega \in \mathfrak{S}_D$ iff $\mathcal{C} \subset \mathcal{N}_\omega := \{A \in \mathcal{R}(X,\sigma) \mid \omega(A^*A) = 0\}$, where $\mathcal{C} := \{i\lambda R(\lambda,f) - \mathbb{1} \mid f \in C,\ \lambda \in \mathbb{R}\backslash 0\}$. Note that $\mathcal{C}^* = \mathcal{C}$.

8.1 Proposition *Given the data above, we have:*

(i) $\mathfrak{S}_D = \{\omega \in \mathfrak{S}(\mathcal{R}(X,\sigma)) \mid \omega(R(1,f)) = -i,\ f \in C\}$.

(ii) If $\omega \in \mathfrak{S}_D$*, then it is not regular. In particular, if* $\sigma(g,C) \neq 0$ *for some* $g \in X$*, then* $\pi_\omega(R(\lambda,g)) = 0$ *for all* $\lambda \in \mathbb{R}\backslash 0$.

(iii) $\mathfrak{S}_D \neq \emptyset$ *iff* $\sigma(C,C) = 0$.

Henceforth we will assume that $\sigma(C,\,C) = 0$ and hence $\mathfrak{S}_D \neq \emptyset$. We examine the algebraic structures produced by these constraints, cf. [12,16]. For the left ideal generated by the constraints we have $\mathcal{N} := [\mathcal{R}(X,\sigma)\mathcal{C}] = \bigcap\,\{\mathcal{N}_\omega \mid \omega \in \mathfrak{S}_D\}$, cf. Theorem 3.13.5 in [23], where $[\,\cdot\,]$ denotes the closed span of its argument. Then there is the following structure, cf. [16].

8.2 Proposition

Let $\mathcal{D} := \mathcal{N}\bigcap\mathcal{N}^*$ *and* $\mathcal{O} := \{F \in \mathcal{R}(X,\sigma) \mid [F,\,\mathcal{D}] \subset \mathcal{D}\}$*. Then*

(i) $\mathcal{D} = \mathcal{N}\bigcap\mathcal{N}^*$ *is the unique maximal* C**–algebra in* $\bigcap\,\{\mathrm{Ker}\,\omega \mid \omega \in \mathfrak{S}_D\}$*. Moreover* $\mathcal{D}$ *is a hereditary* C**–subalgebra of* $\mathcal{R}(X,\sigma)$.

(ii) $\mathcal{O} = \{F \in \mathcal{R}(X,\sigma) \mid F\mathcal{D} \subset \mathcal{D} \supset \mathcal{D}F\}$*, i.e. it is the relative multiplier algebra of* $\mathcal{D}$ *in* $\mathcal{R}(X,\sigma)$.

(iii) $\mathcal{O} = \{F \in \mathcal{R}(X,\sigma) \mid\ [F,\,\mathcal{C}] \subset \mathcal{D}\}$.

(iv) $\mathcal{D} = [\mathcal{O}\,\mathcal{C}] = [\mathcal{C}\,\mathcal{O}]$.

Then $\mathcal{O}$ is the C*–algebraic analogue of Dirac's observables (the weak commutant of the constraints), and the algebra of physical observables is $\mathcal{P} := \mathcal{O}/\mathcal{D}$, using the fact that $\mathcal{D}$ is a closed two–sided ideal of $\mathcal{O}$. Note that by (iii) the relative commutant $\mathcal{C}'$ of $\mathcal{C}$ in $\mathcal{R}(X,\sigma)$ (traditionally regarded as algebra of observables) is contained in $\mathcal{O}$.

Having introduced the general concepts, let us determine these algebras more explicitly. If a $g \in X$ satisfies $\sigma(g,C) = 0$, then $R(\mu,g) \in \mathcal{C}' \subset \mathcal{O}$. On the other hand, if $\sigma(g,C) \neq 0$, then by Proposition 8.1(ii) we get that $\pi_\omega(R(\mu,g)) = 0$ for all $\omega \in \mathfrak{S}_D$. But then $AR(\mu,g)B \in \mathcal{N}_\omega\bigcap\mathcal{N}_\omega^*$ for all $\omega \in \mathfrak{S}_D$ and $A,\,B \in \mathcal{R}(X,\sigma)$, *i.e.* $AR(\mu,g)B \in \mathcal{D} \subset \mathcal{O}$. Thus all the generating elements of $\mathcal{R}(X,\sigma)$ are in $\mathcal{O}$. As $\mathcal{O}$ is a C*–algebra it follows that $\mathcal{O} = \mathcal{R}(X,\sigma)$, and hence that $\mathcal{D}$ is a proper ideal of $\mathcal{R}(X,\sigma)$. Then by Proposition 8.2(iv) we can write $\mathcal{D} = [\mathcal{R}(X,\sigma)\mathcal{C}] = [\mathcal{C}\mathcal{R}(X,\sigma)]$. Moreover, as any monomial in the resolvents containing a resolvent not in $\mathcal{C}'$ is in $\mathcal{D}$, we conclude that any $A \in \mathcal{R}(X,\sigma)$ can be approximated in norm by elements in $\mathcal{C}'$ modulo elements of $\mathcal{D}$. Thus $\mathcal{P} = \mathcal{C}'/(\mathcal{C}'\bigcap\mathcal{D})$. Thus, if π_ω is the GNS–representation of a Dirac state $\omega \in \mathfrak{S}_D$, then $\pi_\omega(\mathcal{R}(X,\sigma))$ is a homomorphic image of the traditional observables $\mathcal{C}'$. To summarize, we have shown:

8.3 Proposition

$\mathcal{O} = \mathcal{R}(X,\sigma)$ *with the proper ideal* $\mathcal{D} = [\mathcal{R}(X,\sigma)\mathcal{C}] = [\mathcal{C}\mathcal{R}(X,\sigma)]$*, and* $\mathcal{P} = \mathcal{C}'/(\mathcal{C}'\bigcap\mathcal{D})$.

So Dirac constraining of linear bosonic constraints is considerably simpler in the resolvent algebra $\mathcal{R}(X,\sigma)$ than in the CCR–algebra $\overline{\Delta(X,\,\sigma)}$ cf. [12].

9 Discussion

Starting from the basic relations of the canonical observables in quantum physics in resolvent form, we were led in a natural manner to an intriguing mathematical structure, the resolvent algebra. This C*–algebra has many desirable features for the treatment of finite and infinite dimensional quantum systems: It allows for the formulation of physically relevant dynamical laws, it contains a multitude of physically significant observables and it provides a powerful framework for the analysis of physical states. Moreover, it is a convenient setting for the discussion of singular representations of the canonical observables which appear naturally in the discussion of quantum constraints and in the modelling of Fermionic symmetries [5] in the context of supersymmetry or of BRST–constraint theory.

The modest price for these conveniences is the fact that the resolvent algebra has a non–trivial ideal structure (cf. the remark at the end of the proof of Proposition 6.1). These ideals correspond to the kernels of representations of the resolvent algebra in which some of the underlying fields have "infinite values" (and thus become physically meaningless). The fact that the resolvents of these fields simply disappear in the respective representations makes the representation theory of the resolvent algebra particularly simple (cf. Proposition 4.7). On the other hand, the physically significant regular representations of the resolvent algebra are faithful, and are thus in perfect agreement with the principle of physical equivalence [18].

The resolvent algebra competes with several other approaches to the treatment of canonical quantum systems, such as the Weyl algebra, its twisted convolution form and other possible variants, cf. [19]. Surprisingly, its existence seems to have escaped observation so far; for both from a conceptual and technical point of view the resolvent algebra seems superior to these other approaches. We could only illustrate here some of its many advantages, but our results suggest that further study and applications of the resolvent algebra to concrete quantum systems are worthwhile.

10 Proofs

Proof of Proposition 2.1

Assume that $M \in \overline{\Delta(X, \sigma)}$ is nonzero such that $\phi(f)M$ is bounded for some nonzero $f \in X$. Let $W_t := \exp(it\phi(f))$, $t \in \mathbb{R}$ and denote the spectral resolution of $\phi(f)$ by $\phi(f) = \int \lambda \, dP(\lambda)$, then

$$\begin{aligned} \|(W_t - \mathbb{1})M\| &= \Big\| \int (e^{it\lambda} - 1) dP(\lambda) M \Big\| \\ &= |t| \Big\| \int \frac{(e^{it\lambda} - 1)}{t\lambda} \, dP(\lambda) \int \lambda' \, dP(\lambda') M \Big\| \\ &\leq |t| \|\phi(f)M\| \longrightarrow 0 \end{aligned}$$

as $t \to 0$, where we used the bound $|\frac{e^{ix}-1}{x}| \leq 1$. Let $\mathcal{J} \subset \overline{\Delta(X, \sigma)}$ consist of all elements M such that $\|(W_t - \mathbb{1})M\| \to 0$ as $t \to 0$. This is clearly a norm-closed linear space, and by the inequality $\|(W_t - \mathbb{1})MA\| \leq \|(W_t - \mathbb{1})M\| \, \|A\|$ it is also a right ideal. To see that it is a two sided ideal

note that

$$\left\|(W_t - \mathbb{1})e^{i\phi(g)}M\right\| = \left\|(W_t e^{it\sigma(f,g)} - \mathbb{1})M\right\|$$

still converges to 0 as $t \to 0$, and use the fact that $\overline{\Delta(X,\sigma)}$ is the norm closure of the span of $\{\, e^{i\phi(g)} \mid g \in X \,\}$. As $\overline{\Delta(X,\sigma)}$ is simple and $\|W_t - 1\| = 2$, $t \neq 0$, it follows that $\mathcal{J} \ni M$ is zero.

Proof of Proposition 3.3

(i) Using $R(\lambda, f)^* = R(-\lambda, f)$ and the C*–property of the norm, we get

$$2|\lambda|\|\pi(R(\lambda,f))\|^2 = \|\pi(2\lambda R(\lambda,f)R(\lambda,f)^*)\| = \|\pi(R(\lambda,f) - R(\lambda,f)^*)\| \leq 2\|\pi(R(\lambda,f))\|\,.$$

Thus we get that either $\pi(R(\lambda,f)) = 0$ which implies $\|\pi(R(\lambda,f))\| \leq 1/|\lambda|$, or $\pi(R(\lambda,f)) \neq 0$ in which case a cancellation gives $\|\pi(R(\lambda,f))\| \leq 1/|\lambda|$. Since $\mathcal{R}_0$ is generated polynomially by the elements $R(\lambda,f)$, it follows that for each $A \in \mathcal{R}_0$ there is a $c_A \geq 0$ such that $\|\pi(A)\| \leq c_A$ for all Hilbert space representations π of $\mathcal{R}_0$.
(ii) Let $\mathcal{N}_\omega := \{A \in \mathcal{R}_0 \mid \omega(A^*A) = 0\}$, then the image of the factor map $\xi : \mathcal{R}_0 \to \mathcal{R}_0/\mathcal{N}_\omega$ is a pre–Hilbert space, equipped with the inner product $(\xi(A), \xi(B)) := \omega(A^*B)$. The Hilbert closure of $\mathcal{R}_0/\mathcal{N}_\omega$ is the GNS–space $\mathcal{H}_\omega$. The GNS–representation is defined on $\mathcal{R}_0/\mathcal{N}_\omega$ by $\pi_\omega(A)\,\xi(B) = \xi(AB)$ which is well–defined because $\mathcal{N}_\omega$ is a left ideal. It is clear that this is a *-representation on the dense invariant domain $\mathcal{R}_0/\mathcal{N}_\omega \subset \mathcal{H}_\omega$ and that $\xi(\mathbb{1}) =: \Omega_\omega$ is a cyclic vector for it.
Now by Eq. (7) we have for $\Psi \in \mathcal{R}_0/\mathcal{N}_\omega$ that

$$\begin{aligned}
\|\pi_\omega(R(\lambda,f))\Psi\|^2 &= (\pi_\omega(R(\lambda,f))\Psi,\, \pi_\omega(R(\lambda,f))\Psi) = (\Psi,\, \pi_\omega(R(\lambda,f)^*R(\lambda,f))\Psi) \\
&= \left|(2i\lambda)^{-1}(\Psi,\, [\pi_\omega(R(\lambda,f)) - \pi_\omega(R(\lambda,f))^*]\Psi)\right| \\
&\leq |\lambda|^{-1}\|\pi_\omega(R(\lambda,f))\Psi\| \cdot \|\Psi\|
\end{aligned}$$

by the Cauchy–Schwartz inequality. Thus $\|\pi_\omega(R(\lambda,f))\| \leq |\lambda|^{-1}$ and so π_ω is bounded.

Proof of Theorem 3.6

(i) By (4) we have that $i(\mu - \lambda)R(\lambda,f)R(\mu,f) = R(\lambda,f) - R(\mu,f) = -(R(\mu,f) - R(\lambda,f)) = i(\mu - \lambda)R(\mu,f)R(\lambda,f)$, *i.e.* $[R(\lambda,f),\, R(\mu,f)] = 0$.
(ii) This follows directly from Eq. (5) by interchanging λ and f with μ and g resp.
(iii) The Fock representation π defines a bounded representation of $\mathcal{R}_0$, where the resolvents of the fields give the Fock representation induced on $\mathcal{R}(X,\sigma)$, *i.e.* $\pi(R(\lambda,f)) = (i\lambda - \phi_\pi(f))^{-1}$. Since this is bounded, it defines a unique representation of $\mathcal{R}(X,\sigma)$. Now

$$\|R(\lambda,f)\| \geq \|\pi(R(\lambda,f))\| = \|(i\lambda - \phi_\pi(f))^{-1}\| = \sup_{t\in\sigma(\phi_\pi(f))}\left|\frac{1}{i\lambda - t}\right| = \frac{1}{|\lambda|}$$

using the fact that the spectrum $\sigma(\phi_\pi(f)) = \mathbb{R}$, cf. for example Chapter 3.1 in [9], in particular the proof of Theorem 5. Thus by Proposition 3.3(i) one arrives at $\|R(\lambda,f)\| = 1/|\lambda|$.
(iv) Rearrange Eq. (4) to get:

$$R(\lambda,f)(\mathbb{1} - i(\lambda_0 - \lambda)R(\lambda_0,f)) = R(\lambda_0,f)\,.$$

Now by (iii), if $|\lambda_0 - \lambda| < |\lambda_0|$ then $\|i(\lambda_0 - \lambda)R(\lambda_0, f)\| < 1$, and hence $(\mathbb{1} - i(\lambda_0 - \lambda)R(\lambda_0, f))^{-1}$ exists, and is given by a norm convergent power series in $i(\lambda_0 - \lambda)R(\lambda_0, f)$. That is, we have that

$$R(\lambda, f) = R(\lambda_0, f)(\mathbb{1} - i(\lambda_0 - \lambda)R(\lambda_0, f))^{-1} = \sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n\, i^n R(\lambda_0, f)^{n+1}$$

when $|\lambda_0 - \lambda| < |\lambda_0|$, as claimed.
(v) The map $\alpha_T(R(\lambda, f)) = R(\lambda, Tf)$ permutes the generating elements of the free unital *-algebra generated by $\{R(\lambda, f) \mid \lambda \in \mathbb{R}\backslash 0,\ f \in X\}$, hence defines an automorphism of it. Since T is symplectic, this automorphism preserves the relations (1) to (6), hence the ideal they generate in the free algebra, hence it factors through to an automorphism of $\mathcal{R}_0$. Moreover since α is a *–automorphism, it maps $\mathfrak{S}$ to itself bijectively, hence it preserves the enveloping C*–seminorm, and so defines an automorphism of $\mathcal{R}(X, \sigma)$.

Proof of Proposition 3.7

The argument is similar to the last part of the preceding proof: The map $\beta_h(R(z, f)) := R(z + ih(f),\, f)$ permutes the generating elements of the free unital *-algebra generated by the set $\{\, R(z, f) \mid z \in \mathbb{C}\backslash i\mathbb{R},\ f \in X \,\}$ hence defines an automorphism of it. This automorphism preserves the relations (8) to (13), hence the ideal they generate in the free algebra, hence it factors through to an automorphism of the *–algebra obtained from factoring out this ideal. Since β_h is a *–automorphism, it maps $\mathfrak{S}$ to itself bijectively, hence preserves the enveloping C*–seminorm, and hence it defines an automorphism of $\mathcal{R}(X, \sigma)$.

Proof of Theorem 3.8

Since the monomials $\prod_{j=1}^{k} R(\lambda_j, f_j)$ span a dense subspace of $\mathcal{R}(X, \sigma)$, we get that the right ideal $[R(\lambda, f)\mathcal{R}(X, \sigma)]$ is the closed span of the monomials $R(\lambda, f) \prod_{j=1}^{k} R(\lambda_j, f_j)$. But by Eq. (6)

$$R(\lambda, f)\, R(\mu, g) = R(\mu, g)\, R(\lambda, f) + i\sigma(f, g)\, R(\lambda, f)\, R(\mu, g)^2 R(\lambda, f) \in \mathcal{R}(X, \sigma)R(\lambda, f)$$

so in each monomial we can progressively move $R(\lambda, f)$ past all the $R(\lambda_j, f_j)$ until it stands on the right, and hence $[R(\lambda, f)\mathcal{R}(X, \sigma)] \subseteq [\mathcal{R}(X, \sigma)R(\lambda, f)]$. Likewise, by using Eq. (6) to move $R(\lambda, f)$ to the left, we get the reverse inclusion and hence equality $[R(\lambda, f)\mathcal{R}(X, \sigma)] = [\mathcal{R}(X, \sigma)R(\lambda, f)]$. By a similar argument we also get $[R(\lambda, f)\mathcal{R}(X, \sigma)] = [\mathcal{R}(X, \sigma)R(\lambda, f)\mathcal{R}(X, \sigma)]$ which is obviously the closed two–sided ideal generated by $R(\lambda, f)$.

Next, we need to prove that $[\mathcal{R}(X, \sigma)R(\lambda, f)]$ is proper. If it is not proper, then $\mathbb{1} \in [\mathcal{R}(X, \sigma)R(\lambda, f)]$ and hence there is a sequence $\{A_n\} \subset \mathcal{R}(X, \sigma)$ such that $A_n R(\lambda, f) \to \mathbb{1}$ in norm. Let π be the Fock representation on $\mathcal{H}_\pi$, then $\phi_\pi(f) := i\mathbb{1} - \pi(R(1, f))^{-1}$ is selfadjoint and $\pi(R(\lambda, f)) = \int_{\mathbb{R}} (i\lambda - t)^{-1}\, dP(t)$ where dP indicates the spectral measure of $\phi_\pi(f)$. Choose $\Psi_k \in P[k,\, k+1]\mathcal{H}_\pi$ with $\|\Psi_k\| = 1$, then since the spectrum of $\phi_\pi(f)$ is $\mathbb{R}$,

$$\|\pi(R(\lambda, f))\, \Psi_k\| \leq \sup_{t \in [k, k+1]} \left|(i\lambda - t)^{-1}\right| = (\lambda^2 + k^2)^{-1/2}\,. \tag{19}$$

Now $\pi(A_n R(\lambda, f)) \to \mathbb{1}$ uniformly as $n \to \infty$, so for $\varepsilon < 1$ there is an N such that for all $k > 0$, $\|\pi(A_n R(\lambda, f))\Psi_k\| \geq 1 - \varepsilon$. By Eq. (19) $\|\pi(A_n R(\lambda, f))\Psi_k\| \leq \|\pi(A_n)\|/\sqrt{\lambda^2 + k^2}$ and consequently $\|\pi(A_n)\| \geq (1-\varepsilon)\sqrt{\lambda^2 + k^2}$. As $\varepsilon < 1$ and k arbitrary we conclude that $\pi(A_n)$ is unbounded. But this is a contradiction, hence $[\mathcal{R}(X,\sigma)R(\lambda, f)]$ is proper.

For the last statement, consider the intersection $[R(\lambda_1, f_1)\mathcal{R}(X,\sigma)] \cap [R(\lambda_2, f_2)\mathcal{R}(X,\sigma)] \ni A$. Since $A \in [R(\lambda_1, f_1)\mathcal{R}(X,\sigma)]$ there is a sequence $\{B_n\} \subset \mathcal{R}(X,\sigma)$ such that $R(\lambda_1, f_1)\, B_n \to A$ in norm. Let $\{E_\iota\}$ be an approximate identity of $[R(\lambda_2, f_2)\mathcal{R}(X,\sigma)]$ then we can construct a sequence $R(\lambda_1, f_1)\, B_n\, E_{\iota_n} \to A$ in norm, using the fact that $A \in [R(\lambda_2, f_2)\mathcal{R}(X,\sigma)]$. Since $\{E_\iota\} \subset [R(\lambda_2, f_2)\mathcal{R}(X,\sigma)]$ we can find $F_n \in \mathcal{R}(X,\sigma)$ such that $F_n R(\lambda_2, f_2)$ are arbitrarily close to E_{ι_n} and hence we can find such F_n such that the sequence $R(\lambda_1, f_1)\, B_n\, F_n\, R(\lambda_2, f_2) \to A$ in norm, and hence $A \in [R(\lambda_1, f_1)\, R(\lambda_2, f_2)\mathcal{R}(X,\sigma)]$. Since it is trivial that

$$[R(\lambda_1, f_1)\, R(\lambda_2, f_2)\mathcal{R}(X,\sigma)] \subseteq [R(\lambda_1, f_1)\mathcal{R}(X,\sigma)] \cap [R(\lambda_2, f_2)\mathcal{R}(X,\sigma)]$$

the equality now follows. For more intersections we repeat an inductive version of the argument.

Proof of Theorem 4.1

(i) Let $K := \operatorname{Ker} \pi(R(\lambda, f))$ and note that from Eq. (5)

$$\pi\left(R(\lambda, f)\, R(\mu, g)\right) = \pi\left(R(\mu, g) + i\sigma(f, g)\, R(\lambda, f)\, R(\mu, g)^2\right)\, \pi(R(\lambda, f))$$

hence $\pi\left(R(\lambda, f)\, R(\mu, g)\right)\, K = 0$ and so $\pi(R(\mu, g))\, K \subseteq K$ for all μ and g. Since $R(\mu, g)^* = R(-\mu, g)$, the subspace K reduces $\pi(\mathcal{R}(X,\sigma))$. Put $\pi_1 := \pi \restriction K$ and $\pi_2 := \pi \restriction K^\perp$ to obtain the desired decomposition. Uniqueness is clear.

(ii) According to (i) $\pi = \pi_1 \oplus \pi_2$ where $\pi_1(R(\lambda, f)) = 0$ and $\operatorname{Ker} \pi_2(R(\lambda, f)) = \{0\}$. Thus by Theorem 4.2(ii), to be proved subsequently, we obtain $\operatorname*{s-lim}_{\lambda\to\infty} i\lambda\pi(R(\lambda, f)) = \mathbb{1}_{K^\perp} =: P_f \in \pi(\mathcal{R}(X,\sigma))''$, and it is obvious that P_f commutes with $\pi(\mathcal{R}(X,\sigma))''$. Since $\pi_2(R(\lambda, f)))$ is invertible, it has dense range, so the closure of its range is $K^\perp$ and the projection on this is P_f. Moreover

$$\pi\left([R(\lambda, f)\, \mathcal{R}(X,\sigma)]\right) \mathcal{H} = \pi_2\left([R(\lambda, f)\, \mathcal{R}(X,\sigma)]\right) K^\perp$$

and this is dense in $K^\perp$ because $R(\lambda, f) \in [R(\lambda, f)\, \mathcal{R}(X,\sigma)]$.

(iii) If π is factorial, its center is trivial, so its central projections can only be 0 or $\mathbb{1}$.

(iv) By Theorem 3.8, $[R(\lambda, f)\mathcal{R}(X,\sigma)]$ is a proper ideal of $\mathcal{R}(X,\sigma)$, so any state of the factor algebra $\mathcal{R}(X,\sigma)/[R(\lambda, f)\mathcal{R}(X,\sigma)]$ lifts to a state ω of $\mathcal{R}(X,\sigma)$ with $R(\lambda, f) \in [R(\lambda, f)\mathcal{R}(X,\sigma)]$ in its kernel. On the other hand, if we are given a state ω of $\mathcal{R}(X,\sigma)$ with $R(\lambda, f) \in \operatorname{Ker}\omega$ then also $R(\lambda, f)^* \in \operatorname{Ker}\omega$ and hence by $R(\lambda, f) - R(\lambda, f)^* = -2i\lambda R(\lambda, f)R(\lambda, f)^*$ we get that $R(\lambda, f)R(\lambda, f)^* \in \operatorname{Ker}\omega$, *i.e.* $R(\lambda, f) \in \mathcal{N}_\omega^*$ which is a right ideal. Thus $[R(\lambda, f)\mathcal{R}(X,\sigma)] \subset \mathcal{N}_\omega^* \subset \operatorname{Ker}\omega$, and as $[R(\lambda, f)\mathcal{R}(X,\sigma)]$ is a two–sided ideal it must be in $\operatorname{Ker}\pi_\omega$.

Proof of Theorem 4.2

(i) Observe that by Theorem 1 in [30, p 216], we deduce from $\operatorname{Ker}\pi(R(1, f)) = \{0\}$ that $\pi(R(\lambda, f))$ is the resolvent of $\phi_\pi(f)$, *i.e.* we have now for all $\lambda \neq 0$ that $\phi_\pi(f) = i\lambda\mathbb{1} - \pi(R(\lambda, f)))^{-1}$. Then

$$\phi_\pi(\mu f) \;=\; i\mathbb{1} - \pi(R(1, \mu f))^{-1} = i\mathbb{1} - \mu\pi(R(\tfrac{1}{\mu}, f))^{-1}$$

$$= \mu\left(i\tfrac{1}{\mu}\mathbb{1} - \pi(R(\tfrac{1}{\mu}, f))^{-1}\right) = \mu\,\phi_\pi(f)\,.$$

Thus

$$\begin{aligned}
\phi_\pi(f)^* &= \left(i\mathbb{1} - \pi(R(1,f))^{-1}\right)^* \supseteq -i\mathbb{1} - \left(\pi(R(1,f))^{-1}\right)^* \\
&= -i\mathbb{1} - \pi(R(1,f)^*)^{-1} = -i\mathbb{1} - \pi(R(-1,f))^{-1} \\
&= -i\mathbb{1} + \pi(R(1,-f))^{-1} = -\phi_\pi(-f) = \phi_\pi(f)
\end{aligned}$$

and hence $\phi_\pi(f)$ is symmetric. To see that it is selfadjoint note that:

$$\mathrm{Ran}\ (\phi_\pi(f) \pm i\mathbb{1}) = \mathrm{Ran}\ \left(-\pi(R(\pm 1, f))^{-1}\right) = \mathrm{Dom}\ (\pi(R(\pm 1, f))) = \mathcal{H}_\pi$$

hence the deficiency spaces $(\mathrm{Ran}\ (\phi_\pi(f) \pm i\mathbb{1}))^\perp = \{0\}$ and so $\phi_\pi(f)$ is selfadjoint.
For the domain claim, recall that $\mathrm{Dom}\,\phi_\pi(f) = \mathrm{Ran}\,\pi(R(1,f))$. So

$$\begin{aligned}
\pi(R(\lambda,f))\mathrm{Dom}\,\phi_\pi(h) &= \pi(R(\lambda,f))\pi(R(1,h))\mathcal{H}_\pi \\
&= \pi\left(R(1,h)R(\lambda,f) + i\sigma(f,h)R(1,h)R(\lambda,f)^2R(1,h)\right)\mathcal{H}_\pi \\
&\subseteq \pi\left(R(1,h)\right)\mathcal{H}_\pi = \mathrm{Dom}\,\phi_\pi(h).
\end{aligned}$$

(ii) Let $\phi_\pi(f) = \int \mu\, dP(\mu)$ be the spectral resolution of $\phi_\pi(f)$. Then $\pi(R(\lambda,f)) = \int \frac{1}{i\lambda - \mu} dP(\mu)$ hence

$$i\lambda\pi(R(\lambda,f))\Psi = \int \frac{i\lambda}{i\lambda - \mu}\, dP(\mu)\Psi, \quad \Psi \in \mathcal{H}_\pi\,.$$

Since $\left|\frac{i\lambda}{i\lambda-\mu}\right| < 1$ (for $\lambda \in \mathbb{R}\backslash 0$) the integrand is dominated by 1 which is an L^1–function with respect to $dP(\mu)$, and as we have pointwise that $\lim\limits_{\lambda\to\infty} \frac{i\lambda}{i\lambda-\mu} = 1$, we can apply the dominated convergence theorem to get that

$$\lim_{\lambda\to\infty} i\lambda\pi(R(\lambda,f))\Psi = \int dP(\mu)\Psi = \Psi\,.$$

(iii) $i\pi(R(1,\mu f))\Psi = \int \frac{i}{i-\mu\lambda}\, dP(\lambda)\Psi \to \Psi$ as $\mu \to 0$ by the same argument as in (ii).
(iv) Let $\mathcal{D} := \pi(R(1,f)R(1,h))\mathcal{H}_\pi$, then by definition $\mathcal{D} \subseteq \mathrm{Ran}\,\pi(R(1,f)) = \mathrm{Dom}\,\phi_\pi(f)$. Moreover $\pi(R(1,f)R(1,h))\mathcal{H}_\pi = \pi(R(1,h)[R(1,f) + i\sigma(f,h)R(1,f)^2R(1,h)])\mathcal{H}_\pi \subseteq \mathrm{Ran}\,\pi(R(1,h)) = \mathrm{Dom}\,\phi_\pi(h)$, *i.e.* $\mathcal{D} \subseteq \mathrm{Dom}\,\phi_\pi(f) \cap \mathrm{Dom}\,\phi_\pi(h)$. That $\mathcal{D}$ is dense, follows from (iii) of this theorem, using

$$\lim_{\mu\to 0}\lim_{\nu\to 0} \pi(R(1,\mu f)R(1,\nu h))\Psi = -\Psi$$

for all $\Psi \in \mathcal{H}_\pi$, as well as $\mu R(1,\mu f) = R(1/\mu,\ f)$ and the fact mentioned before that all $\pi(R(\lambda,f))$ have the same range for f fixed.

Let $\Psi \in \mathcal{D}$, *i.e.* $\Psi = \pi(R(1,f)R(1,h))\Phi$ for some $\Phi \in \mathcal{H}_\pi$. Then

$$\begin{aligned}
&\pi(R(1,h)R(1,f))[\phi_\pi(f), \phi_\pi(h)]\Psi \\
&= \pi(R(1,h)R(1,f))[\pi(R(1,f))^{-1}, \pi(R(1,h))^{-1}]\pi(R(1,f)R(1,h))\Phi \\
&= \pi(R(1,f)R(1,h) - R(1,h)R(1,f))\Phi = i\sigma(f,h)\pi(R(1,h)R(1,f)^2R(1,h))\Phi \\
&= i\sigma(f,h)\pi(R(1,h)R(1,f))\Psi\,.
\end{aligned}$$

Since $\operatorname{Ker}\pi(R(1,h)R(1,f)) = \{0\}$ it follows that $[\phi_\pi(f), \phi_\pi(h)] = i\sigma(f,h)\mathbb{1}$ on $\mathcal{D}$.
(v) From Eq. (3) we have that

$$\pi(R(\nu,f)) = (i\nu\mathbb{1} - \phi_\pi(f))^{-1} = \tfrac{1}{\nu}\,\pi(R(1, \tfrac{1}{\nu}f)) = \tfrac{1}{\nu}\,\left(i\mathbb{1} - \phi_\pi(\tfrac{1}{\nu}f)\right)^{-1}$$

and hence that $\phi_\pi(f) = \nu\,\phi_\pi(\frac{1}{\nu}f)$, *i.e.* $\phi_\pi(\nu f) = \nu\phi_\pi(f)$ for all $\nu \in \mathbb{R}\backslash 0$ and hence the claim is established for $h = 0$, and we only need to prove it for $\nu = 1$. Consider Eq. (6):

$$\pi\left(R(\lambda,f)R(\mu,h)\right) = \pi\Big(R(\lambda+\mu,\ f+h)[R(\lambda,f) + R(\mu,h) + i\sigma(f,h)R(\lambda,f)^2R(\mu,h)]\Big)$$

and note that as $K := \operatorname{Ker}\pi(R(1,f+h))$ reduces $\pi(\mathcal{R}(X,\sigma))$, it is also in the kernel of the left hand side $\pi\left(R(\lambda,f)R(\mu,h)\right)$ of the equation. However the latter is invertible, hence $K = \{0\}$ (thus the term in square brackets on the rhs is also invertible). It is also clear from the equation that $\operatorname{Dom}\phi_\pi(f+h) = \operatorname{Ran}\pi\left(R(\lambda+\mu,\ f+h)\right) \supset \mathcal{D}$ which is the range of the left hand side; moreover, the invertibility of the term in the square brackets implies that $\mathcal{D}$ is a core for the selfadjoint operator $\phi_\pi(f+h)$.
Next we multiply the equation above on the left by $i(\lambda+\mu)\mathbb{1} - \phi_\pi(f+h)$ and apply this to $(i\mu\mathbb{1} - \phi_\pi(h))(i\lambda\mathbb{1} - \phi_\pi(f))\Psi$, $\Psi \in \mathcal{D}$ to get

$$(i(\lambda+\mu)\mathbb{1} - \phi_\pi(f+h))\Psi = \left((i\mu\mathbb{1} - \phi_\pi(h)) + (i\lambda\mathbb{1} - \phi_\pi(f))\right)\Psi$$

making use of $\left[(i\mu\mathbb{1} - \phi_\pi(h)), (i\lambda\mathbb{1} - \phi_\pi(f))\right]\Psi = i\sigma(h,f)\Psi$. The additivity of ϕ_π on $\mathcal{D}$ then follows.
For the proof that $\pi(R(1,\nu f+h))$ is contained in the von Neumann algebra generated by $\{\pi(R(1,f)),\ \pi(R(1,h))\}$, we begin by showing that the commutants of $\pi(R(\lambda,\nu f))$ coincide for all $\lambda,\nu \in \mathbb{R}\backslash 0$. Since $\pi(R(\lambda,\nu f)) = \frac{1}{\nu}\pi(R(\lambda/\nu, f))$ it suffices to establish this for the case $\nu = 1$. Let $A \in \mathcal{B}(\mathcal{H}_\pi)$ such that $[A, \pi(R(\lambda,f))] = 0$. Then it follows from (4) that

$$[A, \pi(R(\mu,f))]\,(1 + i(\mu-\lambda)\pi(R(\lambda,f))) = [A, \pi(R(\lambda,f))] = 0\,, \qquad \mu \in \mathbb{R}\backslash 0\,.$$

Now the spectrum of $\pi(R(\lambda,f))$ is contained in $\{(i\lambda - x)^{-1} \mid x \in \mathbb{R}\}$, so $(\mathbb{1} + i(\mu-\lambda)\pi(R(\lambda,f)))$ has dense range, hence $[A, \pi(R(\mu,f))] = 0$. The same result clearly holds also for the resolvents $\pi(R(\lambda,\nu h))$, $\lambda,\nu \in \mathbb{R}\backslash 0$. To complete the proof, it therefore suffices to show that the commutant of $\pi(R(\lambda+\mu, f+h))$ contains the commutant of $\{\pi(R(\lambda,f)),\ \pi(R(\mu,h))\}$ for $\lambda,\mu,\lambda+\mu \in \mathbb{R}\backslash 0$. Let $A \in \mathcal{B}(\mathcal{H}_\pi)$ commute with both $\pi(R(\lambda,f))$ and $\pi(R(\mu,h))$. Apply the commutator with A to both sides of the represented Eq. (6) to get:

$$0 = [A,\ \pi\left(R(\lambda+\mu,\ f+h)\right)]\,\pi\Big(R(\lambda,f) + R(\mu,h) + i\sigma(f,h)R(\lambda,f)^2R(\mu,h)\Big)\,.$$

As mentioned above, $\pi(R(\lambda,f) + R(\mu,h) + i\sigma(f,h)R(\lambda,f)^2R(\mu,h))$ is invertible. So, by the preceding results, $0 = [A,\ \pi\left(R(1,\nu f+h)\right)]$, proving $\pi(R(1,\nu f+h)) \in \{\pi(R(1,f)),\ \pi(R(1,h))\}''$.
(vi) From the spectral resolution for $\phi_\pi(f)$ we have trivially that on $\operatorname{Dom}\phi_\pi(f)$

$$\phi_\pi(f)\pi(R(\mu,f)) = \pi(R(\mu,f))\phi_\pi(f) = \int \frac{\lambda}{i\mu - \lambda}dP(\lambda) = i\mu\pi(R(\mu,f)) - \mathbb{1}\,.$$

(vii) Let $\Psi \in \mathrm{Dom}\,\phi_\pi(f) = \mathrm{Ran}\,\pi(R(\lambda, f))$, *i.e.* $\Psi = \pi(R(\lambda, f))\Phi$ for some $\Phi \in \mathcal{H}_\pi$. Then

$$\begin{aligned}
\pi(R(\lambda, f))[\phi_\pi(f),\, \pi(R(\lambda, h))]\Psi &= \pi(R(\lambda, f))[\phi_\pi(f),\, \pi(R(\lambda, h))]\pi(R(\lambda, f))\Phi \\
&= \pi\left([R(\lambda, f), R(\lambda, h)]\right)\Phi = i\sigma(f, h)\pi(R(\lambda, f)R(\lambda, h)^2 R(\lambda, f))\Phi \\
&= i\sigma(f, h)\pi(R(\lambda, f)R(\lambda, h)^2)\Psi\,.
\end{aligned}$$

Since $\mathrm{Ker}\,\pi(R(\lambda, f)) = \{0\}$, it follows that

$$[\phi_\pi(f),\, \pi(R(\lambda, h))] = i\sigma(f, h)\pi(R(\lambda, h)^2)$$

on $\mathrm{Dom}\,\phi_\pi(f)$.

(viii) We first prove the second equality. Let $\Psi, \Phi \in \widetilde{\mathcal{D}} := \mathrm{Span}\,\{\, \chi_{[-a,a]}(\phi_\pi(f))\mathcal{H}_\pi \;\mid\; a > 0\,\}$ where $\chi_{[-a,a]}$ indicates the characteristic function of $[-a, a]$, and note that $\widetilde{\mathcal{D}}$ is a dense subspace. Since $\|\phi_\pi(f)^n \restriction \chi_{[-a,a]}(\phi_\pi(f))\mathcal{H}_\pi\| \leq a^n$ for $n \in \mathbb{N}$, we can use the exponential series, *i.e.*

$$W(f)\Psi := \exp\left(i\phi_\pi(f)\right)\Psi = \sum_{n=0}^{\infty} \frac{(i\phi_\pi(f))^n}{n!}\Psi, \quad \Psi \in \widetilde{\mathcal{D}}\,.$$

By the usual rearrangement of series we then have

$$\big(\Phi,\, W(f)\pi(R(\lambda, h))W(f)^{-1}\Psi\big) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\Phi,\, (\mathrm{ad}\, i\phi_\pi(f))^n(\pi(R(\lambda, h)))\Psi\right)$$

for all $\Phi, \Psi \in \widetilde{\mathcal{D}}$. Using part (vii) repeatedly we have

$$(\mathrm{ad}\, i\phi_\pi(f))^n(\pi(R(\lambda, h)) = n!\, \sigma(h, f)^n\, \pi(R(\lambda, h)^{n+1})$$

and consequently

$$\begin{aligned}
(\Phi, (\mathrm{Ad}\; W(tf))(\pi(R(\lambda, h))\Psi) &= \sum_{n=0}^{\infty} t^n \sigma(h, f)^n (\Phi,\, \pi(R(\lambda, h)^{n+1})\Psi) \\
&= (\Phi,\, \pi(R(\lambda + it\sigma(h, f),\, h))\Psi)
\end{aligned}$$

whenever $|t\sigma(h, f)| < |\lambda|$ and where we made use of the von Neumann series (Theorem 3.6(iii)) in the last step. Since the operators involved are bounded and $\widetilde{\mathcal{D}}$ is dense, it follows that $W(tf)\pi(R(\lambda, h))W(tf)^{-1} = \pi(R(\lambda + it\sigma(h, f),\, h))$ for $|t\sigma(h, f)| < |\lambda|$. By analyticity in λ this can be extended to complex λ such that $\lambda \notin i\mathbb{R}$. Using the group property of $t \mapsto W(tf)$ we then obtain for $\lambda \in \mathbb{R}\backslash 0$ that

$$W(f)\pi(R(\lambda, h))W(f)^{-1} = \pi(R(\lambda + i\sigma(h, f),\, h))\,. \tag{20}$$

To prove the first equation, let us write $W(h)$ in terms of resolvents. Note that $\lim\limits_{n\to\infty}(1 - it/n)^{-n} = e^{it}$, $t \in \mathbb{R}$ and since $\sup\limits_{t\in\mathbb{R}} |(1 - it/n)^{-n}| = 1$, it follows from spectral theory (cf. Theorem VIII.5(d) in [25, p 262]) that

$$W(h) = e^{i\phi_\pi(h)} = \lim_{n\to\infty}\left(1 - i\phi_\pi(h)/n\right)^{-n} = \lim_{n\to\infty} \pi\left(iR(1, -h/n)\right)^n$$

in strong operator topology. Apply Eq. (20) to this to get

$$\begin{aligned} W(f)W(h)W(f)^{-1} &= \operatorname*{s-lim}_{n\to\infty} \pi\left(iR(1+i\sigma(-\tfrac{h}{n},f),-\tfrac{h}{n})\right)^n \\ &= \operatorname*{s-lim}_{n\to\infty} \Big(\mathbb{1}-(i/n)(\sigma(h,f)\mathbb{1}+\phi_\pi(h))\Big)^{-n} \\ &= \exp\left(-i\sigma(f,h)\mathbb{1}+i\phi_\pi(h)\right) = e^{-i\sigma(f,h)}\,W(h)\,. \end{aligned}$$

Making repeatedly use of this equation and Theorem 4.2(v) the asserted Weyl relations then follow by an application of the Trotter product formula, cf. Theorem VIII.31 in [25, p. 297],

$$\begin{aligned} W(f+h) &= \operatorname*{s-lim}_{n\to\infty} \Big(W(\tfrac{1}{n}f)W(\tfrac{1}{n}h)\Big)^n \\ &= \operatorname*{s-lim}_{n\to\infty} e^{i((n^2-n)/2n^2)\,\sigma(f,h)}\,W(f)W(h) = e^{i\sigma(f,h)/2}W(f)W(h)\,. \end{aligned}$$

Finally,

$$W(sf)\mathcal{D} = W(sf)\pi(R(\lambda,f)R(\mu,h))\mathcal{H}_\pi = \pi(R(\lambda,f)R(\mu+i\sigma(h,sf),h))\mathcal{H}_\pi \subset \mathcal{D}$$

hence we conclude that $\mathcal{D}$ is a core for $\phi_\pi(f)$ (cf. Theorem VIII.11 in [25, p 269]).

Proof of Corollary 4.4

Given $\pi \in \operatorname{Reg}(\mathcal{R}(X,\sigma),\mathcal{H})$, then by definition $\widetilde{\pi}$ is regular on $\overline{\Delta(X,\,\sigma)}$. To see that the correspondence $\pi \mapsto \widetilde{\pi}$ is a bijection, we verify that Eq. (15) defines its inverse. This is obvious, because for $\lambda > 0$ one obtains by spectral theory

$$-i\int_0^\infty e^{-\lambda t}\widetilde{\pi}(\delta_{-tf})\,dt = -i\int_0^\infty e^{-\lambda t}\exp\left(-it\phi_\pi(f)\right)dt = \left(i\lambda\mathbb{1}-\phi_\pi(f)\right)^{-1} = \pi(R(\lambda,f))\,,$$

and similarly for $\lambda < 0$. It is also clear from the definition that $\pi \mapsto \widetilde{\pi}$ respects direct sums, and since $\{\exp(i\phi_\pi(f)) \mid f\in X\}'' = \{(i\lambda\mathbb{1}-\phi_\pi(f))^{-1} \mid f\in X,\,\lambda\in\mathbb{R}\backslash 0\}''$ we see that it takes irreducible representations to irreducibles, and conversely. The corresponding bijection $\omega\mapsto\widetilde{\omega}$ for regular states is given by

$$\widetilde{\omega}(A) := (\Omega_\omega,\,\widetilde{\pi}_\omega(A)\Omega_\omega), \quad A\in\overline{\Delta(X,\,\sigma)}$$

from which the claims follow.

Proof of Proposition 4.5

(i) If π is faithful and factorial, then $P_f = \mathbb{1}$ for all $f\in X\backslash 0$, or else $\pi(R(\lambda,f)) = 0$ by Theorem 4.1(iii) for some f which contradicts with the faithfulness of π. But by Theorem 4.1(i), P_f is the projection onto $(\operatorname{Ker}\pi(R(1,f)))^\perp$ so $P_f = \mathbb{1}$ implies $\operatorname{Ker}\pi(R(1,f)) = \{0\}$ and as this holds for all f it follows that π is regular.

(ii) The same argument as in the proof of Theorem 3.6(iii) will apply here because if π is regular, then the spectrum of each $\phi_\pi(f)$ is all of $\mathbb{R}$, and this was the only property of the Fock representation used in the proof of Theorem 3.6(iii).

(iii) $\lim_{\lambda\to\infty} i\lambda\,\omega(R(\lambda,f)C) = (\Omega_\omega,\, \text{s-}\lim_{\lambda\to\infty} i\lambda\,\pi_\omega(R(\lambda,f)C)\Omega_\omega) = (\Omega_\omega, P_f\pi_\omega(C)\Omega_\omega)$ for $C \in \mathcal{R}(X,\sigma)$, where we made use of Theorem 4.1(ii). Since P_f commutes with $\pi_\omega(\mathcal{R}(X,\sigma))$, we see that for $A, B \in \mathcal{R}(X,\sigma)$

$$(\pi_\omega(B)\Omega_\omega, \pi_\omega(A)\Omega_\omega) = \omega(B^*A) = \lim_{\lambda\to\infty} i\lambda\,\omega(R(\lambda,f)B^*A) = (\pi_\omega(B)\Omega_\omega, P_f\pi_\omega(A)\Omega_\omega)\,.$$

This implies that $P_f = \mathbb{1}$, which in turn implies via Theorem 4.1(i) and (ii) that $\pi_\omega(R(\lambda,f))$ is invertible. Since this holds for all $f \in X\backslash 0$ it means that π_ω is regular. The converse is trivial.

Proof of Proposition 4.7

(i) We have $X_R := \{\, f \in X \;\mid\; \operatorname{Ker}\pi(R(1,f)) = \{0\}\,\}$. Let $f,\ g \in X_R$ then by Theorem 4.2(v) we have that $\operatorname{Ker}\pi(R(1,\nu f+g)) = \{0\}$ for $\nu\in\mathbb{R}$, *i.e.* $\nu f + g \in X_R$ and thus X_R is a linear space. Now let $f \in X_S := X\backslash X_R$ and $g \in X_R$. If $f + g \in X_R$ then obviously $f \in X_R$ which is a contradiction, so $f + g \in X_S$.

(ii) Recall that $X_T := \Big\{\, f \in X \;\mid\; \operatorname{Ker}\pi(R(1,f)) = \{0\} \text{ and } \pi(R(1,f))^{-1} \in \mathcal{B}(\mathcal{H}) \Big\} \subset X_R$. Let $f,\ g \in X_T$ then by Theorem 4.2(i) there are selfadjoint operators $\phi_\pi(f)$ and $\phi_\pi(g)$ such that $\pi(R(\lambda,f)) = (i\lambda\mathbb{1} - \phi_\pi(f))^{-1}$ and $\pi(R(\mu,g)) = (i\mu\mathbb{1} - \phi_\pi(g))^{-1}$, and by definition of X_T both field operators are bounded. Then by Theorem 4.2(v) we have that $\phi_\pi(\nu f+g) = \nu\phi_\pi(f) + \phi_\pi(g)$ (hence it is bounded) for $\nu\in\mathbb{R}$, and thus $i\mathbb{1} - \phi_\pi(\nu f+g) = (R(1,\nu f+g))^{-1}$ is bounded, *i.e.* $\nu f + g \in X_T$, and so X_T is a linear space.

Let $f \in X_T$ and $g \in X$ with $\sigma(f,g) \neq 0$. Since $f \in X_T$ we can set $c := \|\pi(R(1,f))^{-1}\|$. Then

$$\|[\pi(R(1,f))^{-1}, \pi(R(1,g))^n]\| \leq 2c\,\|\pi(R(1,g))^n\|\,. \tag{21}$$

On the other hand, since $\operatorname{ad}\pi(R(1,f))^{-1} = -\operatorname{ad}\phi_\pi(f)$ is an inner derivation of $\mathcal{B}(\mathcal{H})$, we obtain from Theorem 4.2(vii)

$$\Big[\pi(R(1,f))^{-1},\, \pi(R(1,g))^n\Big] = -in\,\sigma(f,g)\,\pi(R(1,g)^{n+1})\,.$$

Take the norm of this and use inequality (21) to find for all $n\in\mathbb{N}$ that

$$n\,|\sigma(f,g)|\,\|\pi(R(1,g))^{n+1}\| \leq 2c\,\|\pi(R(1,g))^n\|\,.$$

Now assume that $\pi(R(1,g)) \neq 0$. Then since it is normal (cf. Theorem 3.6(i)) we have:

$$\|\pi(R(1,g))^n\| = \|\pi(R(1,g))\|^n \neq 0\,,$$

and consequently

$$n\,|\sigma(f,g)| \leq 2c\,\|\pi(R(1,g))\|^{-1}, \quad n\in\mathbb{N}\,.$$

However this is impossible because $\sigma(f,g)\neq 0$, hence $\pi(R(1,g)) = 0$.

(iii) If π is factorial, then by Theorem 4.1(iii) for any $f\in X$ the projection onto $\operatorname{Ker}\pi(R(1,f))$ is 0 or $\mathbb{1}$. Since $f\in X_S$ implies that $\operatorname{Ker}\pi(R(1,f)) \neq 0$ it follows that the projection onto the kernel is $\mathbb{1}$, *i.e.* $\pi(R(1,f)) = 0$ for $f\in X_S$.

From part (ii) we have that $X_T \subset X_R^\perp$, and by the preceding step $\pi(R(1,X_S)) = 0$. Thus by

$X = X_R \cup X_S$ it follows that any $\pi(R(1,f))$ with $f \in X_T$ commutes with all $\pi(R(1,g))$, $g \in X$, hence with $\pi(\mathcal{R}(X,\sigma))$. Since π is factorial, its centre is trivial, hence $\pi(R(1,f))$ is a multiple of the identity.

Finally, by (ii) we know that $\sigma(X_T, X_R) = 0$, *i.e.* $X_T \subseteq X_R \cap X_R^\perp$. Conversely, let $h \in X_R \cap X_R^\perp$ then $\pi(R(\lambda,h))$ commutes with $\pi(R(\mu, X_R))$ and as $\pi(R(\lambda, X_S)) = 0$ it commutes with $\pi(\mathcal{R}(X,\sigma))$ and hence since π is factorial $\pi(R(\lambda,h)) \in \mathbb{C}\mathbb{1}\backslash 0$, *i.e.* $h \in X_T$.

(iv) Given the basis $\{q_1, \ldots, q_n\}$ of X_T then by Lemma 11.1(iii) in the Appendix there are conjugates $\{p_1, \ldots, p_n\} \subset X$ which augments it into a symplectic basis of their span Q, and by part (ii) above, all these p_i are in X_S. Obviously Q is nondegenerate. Then by Lemma 11.1(ii) we have the decomposition $X = Q \oplus Q^\perp$ into nondegenerate spaces. Since $X_T \subset Q$ we have that $X_T \cap (Q^\perp \cap X_R) = \{0\}$, *i.e.* $Q^\perp \cap X_R \subset \{0\} \cup (X_R \backslash X_T)$. Now we have the linear decomposition $X_R = (Q^\perp \cap X_R) \dot{+} X_T$, *i.e.* any $f \in X_R$ has a unique decomposition $f = f_1 + f_2$ with $f_1 \in Q^\perp \cap X_R$ and $f_2 \in X_T$. Specifically, we have

$$f_1 := f - \sum_{k=1}^{n} \sigma(p_k, f)\, q_k \in X_R \qquad \text{and} \qquad f_2 := \sum_{k=1}^{n} \sigma(p_k, f)\, q_k \in X_T\,.$$

From (ii) we see that $f_1 \in \{q_1, \ldots, q_k\}^\perp$ and by construction $f_1 \in \{p_1, \ldots, p_k\}^\perp$ and thus $f_1 \in Q^\perp \cap X_R$. We can now show that $Q^\perp \cap X_R$ is nondegenerate. If it is not, then there is a nonzero $h \in Q^\perp \cap X_R$ such that $\sigma(h,\, Q^\perp \cap X_R) = 0$. Then $\sigma(h,\, X_R) = 0$ via the decomposition above, using $\sigma(h,\, X_T) = 0$ by $h \in Q^\perp \subset X_T^\perp$ and $\sigma(h,\, Q^\perp \cap X_R) = 0$. Now for $k \in X_S = X \backslash X_R$ we get $\pi(R(\lambda,k)) = 0$ by part (iii) above, and combining this with the previous fact gives

$$[\pi(R(\lambda,h)),\, \pi(R(\mu,f))] = 0, \quad f \in X\,.$$

Since π is factorial this means that $\pi(R(\lambda,h)) \in \mathbb{C}\mathbb{1}$ which contradicts the fact that $h \in X_R \backslash X_T$. Thus $Q^\perp \cap X_R$ is nondegenerate, and as Q is nondegenerate as well, we have by Lemma 11.1(ii) the decomposition $X = Q \oplus (Q^\perp \cap X_R) \oplus (Q^\perp \cap X_R^\perp)$ into nondegenerate spaces, since the decomposition of X_R above implies that

$$(Q \oplus (Q^\perp \cap X_R))^\perp = (Q + X_R)^\perp = Q^\perp \cap X_R^\perp.$$

Since we have a partition $X = X_R \cup X_S$ and $X_R \subset Q \oplus (Q^\perp \cap X_R)$ we conclude that $(Q^\perp \cap X_R^\perp) \subset X_S \cup \{0\}$.

Proof of Proposition 4.8

(i) Recall that from π_ω we obtain the decomposition in Eq. (16)

$$X = Q \oplus (Q^\perp \cap X_R) \oplus (Q^\perp \cap X_R^\perp)$$

into nondegenerate subspaces. We first show that there is a regular state $\widetilde{\omega}$ which coincides with ω on the *–algebra

$$\text{*–alg}\{R(\lambda,f) \mid f \in Y := Q^\perp \cap X_R,\ \lambda \in \mathbb{R}\backslash 0\}\,.$$

Since Y is nondegenerate we can use Theorem 4.2(viii) to define a representation $\pi_1 : \overline{\Delta(Y,\sigma)} \to \mathcal{B}(\mathcal{H}_\omega)$ by $\pi_1(\delta_f) := \exp(i\phi_{\pi_\omega}(f))$ for $f \in Y$, and hence a regular state $\omega_1(A) := (\Omega_\omega,\, \pi_1(A)\Omega_\omega)$ for $A \in \overline{\Delta(Y,\sigma)}$. From the decomposition $X = Y \oplus Y^\perp$ we obtain the (minimal) tensor product $\overline{\Delta(X,\sigma)} = \overline{\Delta(Y,\sigma)} \otimes \overline{\Delta(Y^\perp,\sigma)}$ [20]. Define the regular state $\widetilde{\omega} := \omega_1 \otimes \omega_2$ on $\overline{\Delta(X,\sigma)}$, where ω_2 is any regular state on $\overline{\Delta(Y^\perp,\sigma)}$. By Corollary 4.4 this corresponds to a regular state $\widetilde{\omega}$ on $\mathcal{R}(X,\sigma)$ (via spectral theory) and it is clear that $\widetilde{\omega}$ coincides with ω on the *–algebra generated by $R(\lambda, Y)$.

Next, we want to construct from $\widetilde{\omega}$ a sequence of regular states $\omega_n := \widetilde{\omega} \circ \gamma_n$ which will converge to ω in the w*–topology, where we now define the automorphisms γ_n.

Start with a basis $\{q_1, \ldots, q_t\}$ of X_T and augment it by $\{p_1, \ldots, p_t\} \subset X_S$ into a symplectic basis of $Q := \mathrm{Span}\{q_1, p_1; \ldots; q_t, p_t\}$, cf. Proposition 4.7(iv). Let $\{q_{t+1}, p_{t+1}; \ldots; q_r, p_r\}$ be a symplectic basis of $Y := Q^\perp \cap X_R$, and let $\{q_{r+1}, p_{r+1}; \ldots; q_s, p_s\}$ be a symplectic basis of $Q^\perp \cap X_R^\perp \subset X_S$. Thus we get a symplectic basis of X:

$$\{q_1, p_1; \ldots; q_t, p_t;\, q_{t+1}, p_{t+1}; \ldots; q_r, p_r;\, q_{r+1}, p_{r+1}; \ldots; q_s, p_s\}$$

which coincides with the decomposition of X above. We decompose the elements

$$\begin{aligned}
f &= \sum_{j=1}^{s}(x_j q_j + y_j p_j) \in X \quad \text{according to} \quad f = f_T + f_{Q\setminus T} + f_R + f_S \quad \text{where:} \\
f_T &:= \sum_{j=1}^{t} x_j q_j \in X_T, \qquad f_{Q\setminus T} := \sum_{j=1}^{t} y_j p_j \in Q\setminus X_T \subset X_S, \\
f_R &:= \sum_{j=t+1}^{r} (x_j q_j + y_j p_j) \in Q^\perp \cap X_R\,, \qquad f_S := \sum_{j=r+1}^{s} (x_j q_j + y_j p_j) \in Q^\perp \cap X_R^\perp \subset X_S\,.
\end{aligned}$$

We define now for $n \in \mathbb{N}$ the symplectic transformation $T_n^{(i)}$ of $f = \sum_{j=1}^{s}(x_j q_j + y_j p_j) \in X$ by

$$T_n^{(i)}(f) = \frac{x_i}{n}\, q_i + n\, y_i\, p_i + \sum_{\substack{j=1\\ j\neq i}}^{s} (x_j q_j + y_j p_j)$$

and the automorphisms $\alpha_n^{(i)} \in \mathrm{Aut}\,\mathcal{R}(X,\sigma)$ by $\alpha_n^{(i)}(R(\lambda, f)) := R(\lambda, T_n^{(i)}(f))$ for $f \in X$, cf. Theorem 3.6(v). Next, using Proposition 3.7 we define the automorphisms $\beta_n \in \mathrm{Aut}\,\mathcal{R}(X,\sigma)$ by $\beta_n(R(\lambda, f)) := R(\lambda + i\sigma(h_n + k, f),\, f)$ for all $f \in X,\ \lambda \in \mathbb{R}\backslash 0$ where

$$h_n := n \sum_{j=r+1}^{s} (n^{j-r} q_j + n^{j+s-2r} p_j) \quad \text{and} \quad k := \sum_{j=1}^{t} b_j p_j$$

with $b_j := (\Omega_\omega, \phi_{\pi_\omega}(q_j)\Omega_\omega)$; note that the fields $\phi_{\pi_\omega}(q_j)$, $j = 1, \ldots t$ are multiples of the identity according to Proposition 4.7(iii). We will make use of the fact that

$$\sigma(h_n + k,\, f) = n \sum_{j=r+1}^{s} (x_j n^{j+s-2r} - y_j n^{j-r}) + \sum_{j=1}^{t} b_j x_j \ \xrightarrow[\infty]{n}\ \pm\infty$$

if any of the coefficients $x_j,\, y_j$ are nonzero for $r+1 \le j \le s$, *i.e.* if $f_S \neq 0$. For in that case $\sigma(h_n + k,\, f)$ is a polynomial in n with degree ≥ 2.

Now define $\gamma_n := \alpha_n^{(1)} \cdots \alpha_n^{(t)} \beta_n \in \operatorname{Aut} \mathcal{R}(X, \sigma)$, so we have

$$\begin{aligned}
\gamma_n(R(\lambda, f)) &= R\Big(\lambda + i\sigma(h_n + k, f),\ \sum_{j=1}^{t} (\frac{x_j}{n}\, q_j + n\, y_j\, p_j) + \sum_{j=t+1}^{s} (x_j q_j + y_j p_j)\Big) \\
&= R\Big(\lambda + i\mu^f + i\lambda_n^f,\ \tfrac{1}{n} f_T + n\, f_{Q\setminus T} + f_R + f_S\Big)
\end{aligned}$$

where $\quad \mu^f \ := \ \sigma(k,\, f) = (\Omega_\omega, \phi_{\pi_\omega}(f_T)\Omega_\omega)\,, \qquad \lambda_n^f := \sigma(h_n,\, f) \xrightarrow[\infty]{n} \begin{cases} \pm\infty & \text{if } f_S \neq 0\,; \\ 0 & \text{if } f_S = 0\,. \end{cases}$

Since γ_n maps resolvents to resolvents, $\pi_{\widetilde{\omega}} \circ \gamma_n$ is also regular, where $\widetilde{\omega}$ is the regular state obtained above. Next, we wish to find the limits $\operatorname*{s-lim}_{n\to\infty} \pi_{\widetilde{\omega}} \circ \gamma_n(R(\lambda, f))$.

• The case $f_{Q\setminus T} \neq 0$. We begin by noting that for any real polynomial ξ_n in n of degree ≥ 2 or for $\xi_n = 0$, $n \in \mathbb{N}$ one has

$$\pi_{\tilde{\omega}}(R(\lambda + i\xi_n,\, n\, f_{Q\setminus T})) = \int \frac{dP(\mu)}{i\lambda - \xi_n - n\mu} \ \xrightarrow[\infty]{n}\ 0 \tag{22}$$

in strong operator topology, where dP is the spectral measure of $\phi_{\pi_{\tilde{\omega}}}(f_{Q\setminus T})$. Now from Eq. (6) we get that

$$\begin{aligned}
&\pi_{\tilde{\omega}}\Big(R(\tfrac{\lambda}{2} + i\mu^f,\, \tfrac{1}{n} f_T + f_R + f_S)\, R(\tfrac{\lambda}{2} + i\lambda_n^f,\, n\, f_{Q\setminus T})\Big) \\
&= \pi_{\tilde{\omega}}\Big(R(\lambda + i\mu^f + i\lambda_n^f,\ \tfrac{1}{n} f_T + n\, f_{Q\setminus T} + f_R + f_S)\, \Big[R(\tfrac{\lambda}{2} + i\mu^f,\, \tfrac{1}{n} f_T + f_R + f_S) + \\
&\quad + R(\tfrac{\lambda}{2} + i\lambda_n^f,\, n\, f_{Q\setminus T}) + i\sigma(f_T, f_{Q\setminus T})\, R(\tfrac{\lambda}{2} + i\mu^f,\, \tfrac{1}{n} f_T + f_R + f_S)^2\, R(\tfrac{\lambda}{2} + i\lambda_n^f,\, n\, f_{Q\setminus T})\Big]\Big)\,.
\end{aligned}$$

If we let $n \to \infty$ in the strong operator topology, and use the fact that multiplication is jointly continuous on bounded sets in the strong operator topology, then by Eq. (22) for $\xi_n = \lambda_n^f$, the left hand side of this is zero, and we get:

$$0 = \operatorname*{s-lim}_{n\to\infty} \pi_{\tilde{\omega}}\Big(R(\lambda + i\mu^f + i\lambda_n^f,\ \tfrac{1}{n} f_T + n\, f_{Q\setminus T} + f_R + f_S\Big)\, R(\tfrac{\lambda}{2} + i\mu^f,\, \tfrac{1}{n} f_T + f_R + f_S)\Big)\,.$$

Consider the last factor. If $f_R + f_S = 0$ then by spectral theory of $\phi_{\pi_{\tilde{\omega}}}(f_T)$ we get that

$$\operatorname*{s-lim}_{n\to\infty} \pi_{\tilde{\omega}}(R(\tfrac{\lambda}{2} + i\mu^f,\, \tfrac{1}{n} f_T)) = R(\tfrac{\lambda}{2} + i\mu^f,\, 0) = \frac{1}{i\lambda/2 - \mu^f}\, \mathbb{1}\,,$$

and hence conclude that

$$0 = \operatorname*{s-lim}_{n\to\infty} \pi_{\tilde{\omega}}\Big(R(\lambda + i\mu^f + i\lambda_n^f,\ \tfrac{1}{n} f_T + n\, f_{Q\setminus T} + f_R + f_S)\Big) = \operatorname*{s-lim}_{n\to\infty} \pi_{\tilde{\omega}} \circ \gamma_n(R(\lambda, f))\,.$$

If $f_R + f_S \neq 0$ then since the resolvent of $\phi_{\pi_{\tilde{\omega}}}(f_T)$ commutes with that of $\phi_{\pi_{\tilde{\omega}}}(f_R + f_S)$ we can use their joint spectral theory:

$$\begin{aligned}
\pi_{\tilde{\omega}}\Big(R(\tfrac{\lambda}{2} + i\mu^f,\, \tfrac{1}{n} f_T + f_R + f_S)\Big) &= \int \frac{dP(\rho, \sigma)}{i\frac{\lambda}{2} - \mu^f - \frac{1}{n}\rho - \sigma} \\
\xrightarrow[\infty]{n} \int \frac{dP(\rho, \sigma)}{i\frac{\lambda}{2} - \mu^f - \sigma} &= \pi_{\tilde{\omega}}(R(\tfrac{\lambda}{2} + i\mu^f,\, f_R + f_S))
\end{aligned}$$

where the limit is in the strong operator topology. Thus we obtain

$$0 = \operatorname*{s-lim}_{n\to\infty} \pi_{\tilde{\omega}}\Big(R(\lambda + i\mu^f + i\lambda_n^f,\ \tfrac{1}{n} f_T + n\, f_{Q\setminus T} + f_R + f_S)\Big)\, \pi_{\tilde{\omega}}(R(\tfrac{\lambda}{2} + i\mu^f,\, f_R + f_S))\,.$$

Since $\pi_{\tilde\omega}$ is regular, the last factor has dense range, hence using uniform boundedness of the resolvents in the first factor, we find again that

$$0 = \operatorname*{s-lim}_{n\to\infty} \pi_{\tilde\omega}\Big(R(\lambda + i\mu^f + i\lambda_n^f,\ \tfrac{1}{n} f_T + n\, f_{Q\setminus T} + f_R + f_S)\Big) = \operatorname*{s-lim}_{n\to\infty} \pi_{\tilde\omega}\circ\gamma_n(R(\lambda, f))\,.$$

• The case $f_{Q\setminus T} = 0$. We have that if $f_T \neq 0$ then

$$\pi_{\tilde\omega}(R(\lambda + i\mu^f + i\lambda_n^f,\ \tfrac{1}{n} f_T)) = \int \frac{dP(\rho)}{i\lambda - \mu^f - \lambda_n^f - \frac{1}{n}\rho} \xrightarrow[\infty]{n} \begin{cases} 0 & \text{if } f_S \neq 0 \\ \frac{1}{i\lambda - \mu^f}\,\mathbb{1} & \text{if } f_S = 0. \end{cases} \tag{23}$$

in strong operator topology, where dP is the spectral measure of $\phi_{\pi_{\tilde\omega}}(f_T)$. When $f_T = 0$ then $\mu^f = 0$ and we get:

$$\pi_{\tilde\omega}(R(\lambda + i\mu^f + i\lambda_n^f,\ \tfrac{1}{n} f_T)) = \frac{1}{i\lambda - \lambda_n^f}\,\mathbb{1} \xrightarrow[\infty]{n} \begin{cases} 0 & \text{if } f_S \neq 0 \\ \frac{1}{i\lambda}\,\mathbb{1} & \text{if } f_S = 0\,. \end{cases}$$

By Eq. (6) we get

$$\begin{aligned} &\pi_{\tilde\omega}\Big(R(\tfrac{\lambda}{2} + i\mu^f + i\lambda_n^f,\ \tfrac{1}{n} f_T)\, R(\tfrac{\lambda}{2},\ f_R + f_S)\Big) \\ &= \pi_{\tilde\omega}\Big(R(\lambda + i\mu^f + i\lambda_n^f,\ \tfrac{1}{n} f_T + f_R + f_S)\,\Big(R(\tfrac{\lambda}{2} + i\mu^f + i\lambda_n^f,\ \tfrac{1}{n} f_T) + R(\tfrac{\lambda}{2},\ f_R + f_S)\Big)\Big)\,. \end{aligned}$$

Now if we let $n \to \infty$ in the strong operator topology, and use Eq. (23) we find that if $f_S \neq 0$ then

$$0 = \operatorname*{s-lim}_{n\to\infty} \pi_{\tilde\omega}\Big(R(\lambda + i\mu^f + i\lambda_n^f,\ \tfrac{1}{n} f_T + f_R + f_S)\Big)\, \pi_{\tilde\omega}\Big(R(\tfrac{\lambda}{2},\ f_R + f_S)\Big)$$

hence since $\pi_{\tilde\omega}$ is regular,

$$0 = \operatorname*{s-lim}_{n\to\infty} \pi_{\tilde\omega}\Big(R(\lambda + i\mu^f + i\lambda_n^f,\ \tfrac{1}{n} f_T + f_R + f_S)\Big) = \operatorname*{s-lim}_{n\to\infty} \pi_{\tilde\omega}\circ\gamma_n(R(\lambda, f))\,.$$

If $f_S = 0$ then

$$\frac{1}{i\frac{\lambda}{2} - \mu^f}\, \pi_{\tilde\omega}\Big(R(\tfrac{\lambda}{2},\ f_R)\Big) = \operatorname*{s-lim}_{n\to\infty} \pi_{\tilde\omega}\Big(R(\lambda + i\mu^f,\ \tfrac{1}{n} f_T + f_R)\,\Big(\frac{1}{i\frac{\lambda}{2} - \mu^f}\,\mathbb{1} + R(\tfrac{\lambda}{2},\ f_R)\Big)\Big)\,.$$

Since the second factor on the right hand side is invertible (as it is a normal operator with continuous spectrum), we obtain via spectral theory of $\phi_{\pi_{\tilde\omega}}(f_R)$ that the equation rearranges to

$$\pi_{\tilde\omega}\Big(R(\lambda + i\mu^f,\ f_R)\Big) = \operatorname*{s-lim}_{n\to\infty} \pi_{\tilde\omega}\Big(R(\lambda + i\mu^f,\ \tfrac{1}{n} f_T + f_R) = \operatorname*{s-lim}_{n\to\infty} \pi_{\tilde\omega}\circ\gamma_n(R(\lambda, f))$$

when $f_R \neq 0$, and the same equation follows trivially when $f_R = 0$. Summarizing the reasoning above, we conclude for a general f that

$$\operatorname*{s-lim}_{n\to\infty} \pi_{\tilde\omega}\circ\gamma_n(R(\lambda, f)) = \begin{cases} 0 & \text{if } f_{Q\setminus T} + f_S \neq 0 \\ \pi_{\tilde\omega}\Big(R(\lambda + i\mu^f,\ f_R)\Big) & \text{if } f_{Q\setminus T} + f_S = 0\ . \end{cases}$$

Hence, for $m \in \mathbb{N}$,

$$\begin{aligned} &\operatorname*{s-lim}_{n\to\infty} \pi_{\tilde\omega}\circ\gamma_n(R(\lambda_{(1)},\ f_{(1)})\cdots R(\lambda_{(m)},\ f_{(m)})) \\ &= \begin{cases} 0 & \text{if } f_{(i)\,Q\setminus T} + f_{(i)\,S} \neq 0 \text{ for any } i, \\ \pi_{\tilde\omega}(R(\lambda_{(1)} + i\mu^{f_{(1)}},\ f_{(1)\,R})\cdots R(\lambda_{(m)} + i\mu^{f_{(m)}},\ f_{(m)\,R})) & \text{if } f_{(i)\,Q\setminus T} + f_{(i)\,S} = 0 \text{ for all } i, \end{cases} \end{aligned}$$

and so

$$\lim_{n\to\infty} \widetilde{\omega}\circ\gamma_n(R(\lambda_{(1)},\,f_{(1)})\cdots R(\lambda_{(m)},\,f_{(m)}))$$
$$= \begin{cases} 0 & \text{if } f_{(i)\,Q\backslash T}+f_{(i)\,S}\neq 0 \text{ for any } i, \\ \widetilde{\omega}\Big(R(\lambda_{(1)}+i\mu^{f_{(1)}},\,f_{(1)\,R})\cdots R(\lambda_{(m)}+i\mu^{f_{(m)}},\,f_{(m)\,R})\Big) & \text{if } f_{(i)\,Q\backslash T}+f_{(i)\,S}=0 \text{ for all } i, \end{cases}$$
$$= \begin{cases} 0 & \text{if } f_{(i)\,Q\backslash T}+f_{(i)\,S}\neq 0 \text{ for any } i, \\ \omega\Big(R(\lambda_{(1)}+i\mu^{f_{(1)}},\,f_{(1)\,R})\cdots R(\lambda_{(m)}+i\mu^{f_{(m)}},\,f_{(m)\,R})\Big) & \text{if } f_{(i)\,Q\backslash T}+f_{(i)\,S}=0 \text{ for all } i \end{cases}$$

since ω coincides with $\widetilde{\omega}$ on *-alg$\{R(z,f)\mid f\in Q^\perp\cap X_R,\ z\in\mathbb{C}\backslash i\mathbb{R}\}$. By Proposition 4.7(iii) we know that $\pi_\omega(R(\lambda,f))=0$ if $f_{Q\backslash T}+f_S\neq 0$ and if $f_{Q\backslash T}+f_S=0$ then

$$\pi_\omega(R(\lambda,f))=[i\lambda\mathbb{1}-\phi_{\pi_\omega}(f_T+f_R)]^{-1}=[(i\lambda-\mu^f)\mathbb{1}-\phi_{\pi_\omega}(f_R)]^{-1}=\pi_\omega(R(\lambda+i\mu^f,\,f_R))$$

by $\phi_{\pi_\omega}(f_T)=\mu_f\mathbb{1}$. Thus

$$\lim_{n\to\infty}\widetilde{\omega}\circ\gamma_n(R(\lambda_{(1)},\,f_{(1)})\cdots R(\lambda_{(m)},\,f_{(m)}))=\omega(R(\lambda_{(1)},\,f_{(1)})\cdots R(\lambda_{(m)},\,f_{(m)}))$$

i.e. $\omega(A)=\lim\limits_{n\to\infty}\widetilde{\omega}\circ\gamma_n(A)$ for all $A\in\mathcal{R}_0$. Since $\mathcal{R}_0$ is norm dense in $\mathcal{R}(X,\sigma)$ the sequence of regular states $\widetilde{\omega}\circ\gamma_n$ converges to the nonregular state ω in the weak *-topology.

(ii) The vector states of the universal regular representation π_r include the regular states $\widetilde{\omega}\circ\gamma_n$ constructed above. Thus from part (i) we see that the given nonregular pure state ω is in the w*-closure of the convex hull of the vector states of π_r and hence by Fell's theorem, (cf. Theorem 1.2 in [11] and [9, p 106]) we find that $\operatorname{Ker}\pi_r\subseteq\operatorname{Ker}\pi_\omega$.

(iii) From part (ii) we see that if $\pi_r(A)=0$ then $\pi_\omega(A)=0$ for all pure states ω of $\mathcal{R}(X,\sigma)$. However according to standard results in the theory of C*–algebras, cf. Lemma 2.3.23 in [4], the norm of $\mathcal{R}(X,\sigma)$ is $\|A\|=\sup\{\omega(A^*A)^{1/2}\mid\omega\in\mathfrak{S}\text{ pure}\}$ and hence $A=0$. Thus π_r is faithful and so $\mathcal{R}(X,\sigma)\cong\mathcal{R}_r(X,\sigma)$.

Proof of Theorem 4.9

(i) On *-alg$\{R(\lambda,f)\mid f\in S,\ \lambda\in\mathbb{R}\backslash 0\}\subset\mathcal{R}(X,\sigma)$ we have the following four norms: $\|\cdot\|_X=$ C*-norm of $\mathcal{R}(X,\sigma)$, $\|\cdot\|_S=$ C*-norm of $\mathcal{R}(S,\sigma)$, and $\|\cdot\|_{\mathrm{reg}S}=$ regular norm of $\mathcal{R}(S,\sigma)$, and $\|\cdot\|_{\mathrm{reg}X}=$ regular norm of $\mathcal{R}(X,\sigma)$. We first prove that $\|A\|_{\mathrm{reg}S}=\|A\|_{\mathrm{reg}X}$ for all A in this *-algebra. Since S is finite dimensional and nondegenerate, we have $X=S\oplus S^\perp$ by Lemma 11.1(ii). By definition of the regular seminorm on $\mathcal{R}(X,\sigma)$ we have

$$\|A\|_{\mathrm{reg}X}=\|\pi_r(A)\|=\sup\{\|\pi_\omega(A)\|\mid\omega\in\mathfrak{S}_r(\mathcal{R}(X,\sigma))\}=\sup\{\sqrt{\omega(A^*A)}\mid\omega\in\mathfrak{S}_r(\mathcal{R}(X,\sigma))\}\,.$$

Thus this will coincide with the regular seminorm of $\mathcal{R}(S,\sigma)$ on *-alg$\{R(\lambda,f)\mid f\in S,\ \lambda\in\mathbb{R}\backslash 0\}$ if we can show that each $\omega\in\mathfrak{S}_r(\mathcal{R}(S,\,\sigma))$ extends to a regular state of $\mathcal{R}(X,\sigma)$. By the bijection of Corollary 4.4 this will be the case if each regular state of $\overline{\Delta(S,\,\sigma)}\subset\overline{\Delta(X,\,\sigma)}$ extends to a regular state of $\overline{\Delta(X,\,\sigma)}$. Now we know by Manuceau [20] that $\overline{\Delta(X,\,\sigma)}=\overline{\Delta(S,\,\sigma)}\otimes\overline{\Delta(S^\perp,\,\sigma)}$, so if ω_1 is a regular state of $\overline{\Delta(S,\,\sigma)}$ then $\omega_1\otimes\omega_2$ will be a regular extension of ω_1 to $\overline{\Delta(X,\,\sigma)}$ if we choose ω_2 to be a regular state of $\overline{\Delta(S^\perp,\,\sigma)}$, which is of course possible. Thus it follows that $\|A\|_{\mathrm{reg}S}=\|A\|_{\mathrm{reg}X}$.

All (regular) states of $\mathcal{R}(X,\sigma)$ restrict to (regular) states of *-alg$\{R(\lambda,f) \mid f\in S,\ \lambda\in\mathbb{R}\backslash 0\}$, so it follows that for all A in this *-algebra:

$$\|A\|_S \geq \|A\|_X \geq \|A\|_{\mathrm{reg}X} = \|A\|_{\mathrm{reg}S}\,.$$

However, since $\|A\|_S = \|A\|_{\mathrm{reg}S}$ by Proposition 4.8(iii), it follows that $\|A\|_S = \|A\|_X$, which establishes the claim. Hence the containment *-alg$\{R(\lambda,f) \mid f\in S,\ \lambda\in\mathbb{R}\backslash 0\}\subseteq\mathcal{R}(X,\sigma)$ extends to an isomorphism of $\mathcal{R}(S,\sigma)$ with a subalgebra of $\mathcal{R}(X,\sigma)$, and we indicate this as containment.
(ii) Since the net $S\to\mathcal{R}(S,\sigma)$ has the partial ordering of containment of the finite dimensional nondegenerate spaces $S\subset X$ (by (i)), and $\mathcal{R}(X,\sigma)$ is generated by all $\mathcal{R}(S,\sigma)$, it is clear that it is the inductive limit of the net.
(iii) From the inductive limit in part (ii), it suffices to verify that π_r restricts to an isomorphism on each $\mathcal{R}(S,\sigma)$, but this holds by $\|A\|_S = \|A\|_{\mathrm{reg}X}$ mentioned above.

Proof of Theorem 4.10

Let $\pi_i:\mathcal{R}(X,\sigma)\to\mathcal{B}(\mathcal{H}_i)$, $i=1,\,2$ be regular representations. Since $\overline{\Delta(X,\,\sigma)}$ is simple, we have a *-isomorphism $\alpha:\pi_1(\overline{\Delta(X,\,\sigma)})\to\pi_2(\overline{\Delta(X,\,\sigma)})$ by $\alpha(\pi_1(A)):=\pi_2(A)$ for all $A\in\overline{\Delta(X,\,\sigma)}$. We want to extend α to $\pi_1(\mathcal{R}(X,\sigma))\subset\pi_1(\overline{\Delta(X,\,\sigma)})''$. Let $S\subset X$ be a finite dimensional nondegenerate subspace, then by the von Neumann uniqueness theorem, both of $\pi_i\restriction\overline{\Delta(S,\,\sigma)}$ are normal to the Fock representation of $\overline{\Delta(S,\,\sigma)}$, hence $\pi_1\restriction\overline{\Delta(S,\,\sigma)}$ is normal to $\pi_2\restriction\overline{\Delta(S,\,\sigma)}$. Then by Theorem 2.4.26 in [3, p 80] we conclude that $\alpha:\pi_1(\overline{\Delta(S,\,\sigma)})\to\pi_2(\overline{\Delta(S,\,\sigma)})$ is normal and extends to a *-homomorphism $\alpha:\pi_1(\overline{\Delta(S,\,\sigma)})''\to\pi_2(\overline{\Delta(S,\,\sigma)})''$ by strong operator continuity. Now $\pi_i(\overline{\Delta(S,\,\sigma)})''\supset\pi_i(\mathcal{R}(S,\sigma))$, and in fact by the Laplace transform (15), for each $A\in\mathcal{R}_0(S)$ there is a sequence $\{A_n\}\subset\overline{\Delta(S,\,\sigma)}$ such that $\pi_i(A_n)\xrightarrow[\infty]{n}\pi_i(A)$ in the strong operator topology. This means that $\alpha\left(\pi_1(\mathcal{R}_0(S))\right)\subseteq\pi_2(\mathcal{R}_0(S))$ and thus α restricts to a *-homomorphism $\alpha:\pi_1(\mathcal{R}(S,\sigma))\to\pi_2(\mathcal{R}(S,\sigma))$ and so

$$\|\pi_2(A)\| = \|\alpha(\pi_1(A))\| \leq \|\pi_1(A)\|\,,\quad A\in\mathcal{R}(S,\sigma)\,.$$

By symmetry of the argument we also get that $\|\pi_1(A)\|\leq\|\pi_2(A)\|$ and hence that $\|\pi_1(A)\| = \|\pi_2(A)\|$ for all $A\in\mathcal{R}(S,\sigma)$. Let the regular representation $\pi_2=\pi_r$ which is faithful by Theorem 4.9(iii), then we have obtained that $\|\pi_1(A)\|=\|A\|$ for all $A\in\mathcal{R}(S,\sigma)$ and for all finite dimensional nondegenerate subspaces $S\subset X$. Since by Theorem 4.9(ii) we know that $\mathcal{R}(X,\sigma)$ is the inductive limit of all the $\mathcal{R}(S,\sigma)$, it follows that π_1 is faithful on all of $\mathcal{R}(X,\sigma)$.

Proof of Proposition 4.13

Note first that if α corresponds to a symplectic transformation, then so does its inverse. Moreover, α and α^{-1} preserve both the set of regular states $\mathfrak{S}_r(\mathcal{R}(X,\sigma))$ and the strongly regular states $\mathfrak{S}_{sr}(\mathcal{R}(X,\sigma))$, respectively. Let $\mathfrak{S}$ be either one of these sets of states and put $\pi_{\mathfrak{S}}:=\bigoplus\limits_{\omega\in\mathfrak{S}}\pi_\omega$. Since both α and α^{-1} preserve $\mathfrak{S}$, one obtains a bijection of $\mathfrak{S}$ by $\omega\mapsto\omega\circ\alpha$. Hence $\pi_{\mathfrak{S}}\circ\alpha$ is just $\pi_{\mathfrak{S}}$ where its direct summands have been permuted. Such a permutation of direct summands can be done by conjugation of a unitary, thus $\pi_{\mathfrak{S}}$ is unitarily equivalent to $\pi_{\mathfrak{S}}\circ\alpha$.

Proof of Theorem 5.1

By Theorem 4.9(i) and (ii) the C*–algebras generated in $\mathcal{R}(X,\,\sigma)$ by $\{R(\lambda, f)\ |\ f\in S,\ \lambda\in\mathbb{R}\backslash 0\}$ and $\{R(\lambda, f)\ |\ f\in S^\perp,\ \lambda\in\mathbb{R}\backslash 0\}$ are $\mathcal{R}(S,\sigma)$ and $\mathcal{R}(S^\perp,\sigma)$. We already have that $\overline{\Delta(X,\,\sigma)} = \overline{\Delta(S,\,\sigma)}\otimes\overline{\Delta(S^\perp,\,\sigma)}$ by Manuceau [20]. Consider a representation $\pi=\pi_1\otimes\pi_2$ of $\overline{\Delta(X,\,\sigma)}$ where π_1 (resp. π_2) is a regular representation of $\overline{\Delta(S,\,\sigma)}$ (resp. $\overline{\Delta(S^\perp,\,\sigma)}$). Then π is regular, hence extends to a representation of $\mathcal{R}(X,\sigma)$ by $\pi(\mathcal{R}(X,\sigma))\subset\pi(\overline{\Delta(X,\,\sigma)})''$ as discussed before, and likewise for π_1 and π_2. Moreover, spectral theory respects tensor products, so if $A\in\mathcal{R}(S,\sigma)$ and $B\in\mathcal{R}(S^\perp,\sigma)$ then $\pi(A)=\pi_1(A)\otimes\mathbb{1}$ and $\pi(B)=\mathbb{1}\otimes\pi_2(B)$ hence $\pi(AB)=\pi(A)\pi(B)=\pi_1(A)\otimes\pi_2(B)$. By Theorem 4.9(iii) we can choose π_1 and π_2 to be faithful, hence

$$\|AB\|\geq\|\pi(AB)\|=\|\pi_1(A)\|\cdot\|\pi_2(B)\|=\|A\|\cdot\|B\|$$

so we conclude that $AB=0$ implies that at least one of A and B must be zero. Since $\mathcal{R}(S,\sigma)$ and $\mathcal{R}(S^\perp,\sigma)$ are commuting subalgebras of $\mathcal{R}(X,\sigma)$ we conclude from this via Exercise 2 in Takesaki [28, p 220], that

$$C^*(\mathcal{R}(S,\sigma)\cup\mathcal{R}(S^\perp,\sigma))\cong\mathcal{R}(S,\sigma)\otimes\mathcal{R}(S^\perp,\sigma)\,.$$

To see that the containment $\mathcal{R}(X,\sigma)\supset C^*(\mathcal{R}(S,\sigma)\cup\mathcal{R}(S^\perp,\sigma))$ is in general proper, we present a simple example. Let $\dim(X)=4$, and choose a symplectic basis $\{q_1,p_1;\,q_2,p_2\}$ and let $S:=\mathrm{Span}\{q_1,p_1\}$, hence $S^\perp=\mathrm{Span}\{q_2,p_2\}$ and $X=S\oplus S^\perp$. Choose a fixed regular state ω on $\mathcal{R}(X,\sigma)$ and define the automorphisms $\beta_n\in\mathrm{Aut}\,\mathcal{R}(X,\sigma)$ by $\beta_n(R(\lambda,f)):=R(\lambda+i\sigma(h_n,f),\,f)$ where $h_n:=n(p_1-p_2)-n^2(q_1-q_2)$, making use of Proposition 3.7. So if $f=x_1q_1+x_2q_2+y_1p_1+y_2p_2$ then $\sigma(h_n,f)=n(x_1-x_2)+n^2(y_1-y_2)$. Thus

$$\begin{aligned}\operatorname*{s-lim}_{n\to\infty}\pi_\omega\circ\beta_n(R(\lambda,f)) &= \operatorname*{s-lim}_{n\to\infty}\pi_\omega(R(\lambda+i\sigma(h_n,f),\,f))\\ &= \operatorname*{s-lim}_{n\to\infty}\int\frac{dP(t)}{i\lambda-\sigma(h_n,f)-t}=\begin{cases}\pi_\omega(R(\lambda,f)) & \text{if } x_1=x_2 \text{ and } y_1=y_2\\ 0 & \text{if } x_1\neq x_2 \text{ or } y_1\neq y_2\end{cases}\end{aligned}$$

where dP is the spectral measure of $\phi_{\pi_\omega}(f)$. Now proceeding as at the end of the proof of Proposition 4.8(i), we conclude that the w*-limit $\widetilde{\omega}:=\lim_{n\to\infty}\omega\circ\beta_n$ defines a state on $\mathcal{R}(X,\sigma)$ such that

$$\widetilde{\omega}(R(\lambda,f))=\begin{cases}\omega(R(\lambda,f)) & \text{if } x_1=x_2 \text{ and } y_1=y_2\\ 0 & \text{if } x_1\neq x_2 \text{ or } y_1\neq y_2\,.\end{cases}$$

Thus $\widetilde{\omega}(R(\lambda,f))=0$ if $f\in S\backslash 0$ or $f\in S^\perp\backslash 0$, and by Theorem 4.1(iv) we have $R(\lambda,f)\in\mathrm{Ker}\,\pi_{\widetilde{\omega}}$ for such f. Since $\mathrm{Ker}\,\pi_{\widetilde{\omega}}$ is a closed two–sided ideal, we get that $C^*(\mathcal{R}(S,\sigma)\cup\mathcal{R}(S^\perp,\sigma))\subset\mathrm{Ker}\,\pi_{\widetilde{\omega}}$. However $\widetilde{\omega}(R(\lambda,\,q_1+q_2+p_1+p_2))=\omega(R(\lambda,\,q_1+q_2+p_1+p_2))\neq 0$, hence $R(\lambda,\,q_1+q_2+p_1+p_2)$ is not an element of $C^*(\mathcal{R}(S,\sigma)\cup\mathcal{R}(S^\perp,\sigma))$ and the containment is proper.

Proof of Theorem 5.3

(i) First, consider the case when $\sigma(f,h)\neq 0$. From Proposition 8.1(i) applied to $C=\{f\}$ we see that there is a state ω such that $\omega(R(1,f))=-i$ (note that the proof of Proposition 8.1 is logically independent from this Theorem). Moreover, by Proposition 8.1(ii) we then

have that $\omega(R(1,h)) = 0$ and thus by Theorem 4.1(iv) we have $\pi_\omega([\mathcal{R}(X,\sigma)R(1,h)]) = 0$ but $\pi_\omega(R(1,f)) \neq 0$, and hence $R(1,f) \notin [\mathcal{R}(X,\sigma)R(1,h)]$.
Next, we prove the claim for the case $\sigma(f,h) = 0$. Augment f, h to a symplectic basis $\{f, p_f;\, h, p_h\}$ using Lemma 11.1(iii) . Let $S := \text{Span}\{f, p_f; h, p_h\} \subset X$ then by Lemma 11.1(ii) we have that $X = S \oplus S^\perp$ and likewise we get that $S = S_1 \oplus S_2$ where $S_1 := \text{Span}\{f, p_f\}$ and $S_2 := \text{Span}\{h, p_h\}$. Then

$$\mathcal{R}(X,\sigma) \supset \mathcal{R}(S_1,\sigma) \otimes \mathcal{R}(S_2,\sigma) \otimes \mathcal{R}(S^\perp,\sigma)$$

by Theorem 5.1 . But then we can choose a product state $\omega = \omega_1 \otimes \omega_2 \otimes \omega_3$ of $\mathcal{R}(S_1,\sigma) \otimes \mathcal{R}(S_2,\sigma) \otimes \mathcal{R}(S^\perp,\sigma)$, such that ω_1 is a Fock state of $\mathcal{R}(S_1,\sigma)$, $\omega_2(R(1,h)) = 0$ (which is possible by Theorem 4.1(iv)), and ω_3 is regular on $\mathcal{R}(S^\perp,\sigma)$. Extend ω by the Hahn–Banach theorem to a state on $\mathcal{R}(X,\sigma)$ (still denoted ω). Then $\omega(R(1,h)) = 0$ and $\omega(R(1,f)) \neq 0$ so the statement now follows as in the preceding step.
(ii) Consider first the case $\sigma(f,h) \neq 0$. Choosing a state ω as in the preceding step we obtain

$$\|R(1,f) - R(1,h)\| \geq |\omega(R(1,f) - R(1,h))| = |-i| = 1\,.$$

For the case $\sigma(f,h) = 0$ we augment (f,h) to a symplectic basis $(f, p_f; h, p_h)$ and put $S := \text{Span}\{f, p_f; h, p_h\} \subset X$. By Theorem 4.9(i) we have the containment $\mathcal{R}(S,\,\sigma) \subset \mathcal{R}(X,\,\sigma)$. Let π be the Schrödinger representation of $\mathcal{R}(S,\,\sigma)$. Since π is regular, by Theorem 4.10 it is faithful, hence applying the joint spectral theory to the two resolvents $\pi(R(1,f))$, $\pi(R(1,h))$ we obtain

$$\|R(1,f) - R(1,h)\| = \|\pi(R(1,f) - R(1,h))\| = \sup_{\rho,\,\sigma\in\mathbb{R}} \left|\frac{1}{i-\rho} - \frac{1}{i-\sigma}\right| = 1\,.$$

(iii) Finally, note that given any f, h as above, we can define $f_\xi := \xi f + (1-\xi)h$, $\xi \in [0,1]$ to obtain an uncountable family such that if $\xi \neq \zeta$ then $f_\xi \notin \mathbb{R} f_\zeta$. Since then the $R(1,f_\xi)$ are all far apart by $\|R(1,f_\xi) - R(1,f_\zeta)\| \geq 1$ it follows that $\mathcal{R}(X,\sigma)$ is nonseparable.

Proof of Theorem 5.4

(i) Since all irreducible regular representations are unitarily equivalent, we may assume that π_0 is the Schrödinger representation on $L^2(\mathbb{R}^n)$ w.r.t. the given symplectic basis. Taking into account the commutation relations (5) of the resolvents, we obtain

$$\pi_0\Big((R(\lambda_1,p_1)R(\mu_1,q_1))\cdots(R(\lambda_n,p_n)R(\mu_n,q_n))\Big) = \prod_{j=1}^n (i\lambda_j - Q_j)^{-1} \cdot \prod_{k=1}^n (i\mu_k - P_k)^{-1}\,,$$

where $Q_j := \phi_{\pi_0}(p_j)$, $P_k := \phi_{\pi_0}(q_k)$ are the familiar position and momentum operators on $L^2(\mathbb{R}^n)$. Now for any pair A, B of continuous, bounded and square integrable functions on $\mathbb{R}^n$ the operator $A(Q_1,\ldots Q_n){\cdot}B(P_1,\ldots P_n)$ is in the Hilbert–Schmidt class cf. [26] Theorem XI.20. Thus the above product of resolvents is in the Hilbert–Schmidt class and hence a compact operator.
(ii) It is well–known that if a C*–algebra acts irreducibly on a Hilbert space, and contains any nonzero element of the compact operators, then it contains all compact operators (cf. [7] Theorem 4.1.10 or [21] Theorem 2.4.9). Thus by (i), $\pi_0(\mathcal{R}(X,\sigma))$ contains all of $\mathcal{K}(\mathcal{H}_0)$, and so,

since π_0 is faithful, $\mathcal{R}(X,\sigma)$ contains an ideal $\mathcal{K}$ isomorphic to $\mathcal{K}(\mathcal{H}_0)$. Uniqueness follows from the fact that up to unitary equivalence (which preserves the compacts), π_0 is unique.
(iii) Since $\mathcal{K}$ is a proper closed two–sided ideal of $\mathcal{R}(X,\sigma)$, each representation $\pi : \mathcal{R}(X,\sigma) \to \mathcal{B}(\mathcal{H}_\pi)$ has a unique decomposition $\pi = \pi_1 \oplus \pi_2$ where π_1 is nondegenerate when restricted to $\mathcal{K}$ and $\pi_2(\mathcal{K}) = 0$. Now if π is regular, then from the products of resolvents in (i), we obtain a sequence $\{I_n\}$ in $\mathcal{K}$ such that $\pi(I_n) \to \mathbb{1}$ in the strong operator topology, cf. Theorem 4.2. Hence π is nondegenerate on $\mathcal{K}$. Conversely, let π be nondegenerate on $\mathcal{K}$. If π is not regular, *i.e.* there is an $f \in X$ such that $\operatorname{Ker}\pi(R(\lambda,f)) \neq 0$, then we know that $\operatorname{Ker}\pi(R(\lambda,f))$ reduces $\pi(\mathcal{R}(X,\sigma))$, *i.e.* we can decompose $\pi = \pi_0 \oplus \pi_R$ where $\pi_0(R(\lambda,f)) = 0$ and $\operatorname{Ker}\pi_R(R(\lambda,f)) = 0$. Since $R(\lambda,f)$ will occur in some products of resolvents in $\mathcal{K}$ as in (i), π_0 when restricted to $\mathcal{K}$ has nonzero kernel. However $\mathcal{K}$ is simple, so $\pi_0(\mathcal{K}) = 0$, and this contradicts the assumption that π is nondegenerate on $\mathcal{K}$. Thus π is regular.
(iv) If $n = 1$ then by (i) we see that if $\sigma(f,g) \neq 0$ then $R(\lambda,f)R(\mu,g) \in \mathcal{K}$. Thus in the factor algebra $\mathcal{R}(X,\sigma)/\mathcal{K}$, all products of noncommuting pairs in the generating set of resolvents will be put to zero, and it is clear that only commuting products survive the factoring. If $n > 1$ then the products $R(\lambda,f)R(\mu,g)$ are generally not in $\mathcal{K}$. Note that if $X = S \oplus S^\perp$ then $\mathcal{R}(S,\sigma)$ imbeds as $\mathcal{R}(S,\sigma) \otimes \mathbb{1}$ of the subalgebra $\mathcal{R}(S,\sigma) \otimes \mathcal{R}(S^\perp,\sigma)$ and so nonzero commutators of elements of $\mathcal{R}(S,\sigma)$ are of the form $B \otimes \mathbb{1}$, which cannot be compact in π_0 for $B \neq 0$.
(v) Let π be a regular representation and let $\{I_n\}$ be a sequence in $\mathcal{K}$ as in (iii). Thus, if $A\mathcal{K} = 0$ then $0 = \operatorname*{s-lim}_{n\to\infty} \pi(AI_n) = \pi(A)$. However π is faithful by Theorem 4.10, hence $A = 0$.
(vi) Since $\mathcal{K}$ is simple, it is clear that it is a minimal nonzero ideal. Let $\mathcal{J} \subset \mathcal{R}(X,\sigma)$ be a nonzero closed two–sided ideal. Then so is $\mathcal{J} \cap \mathcal{K} = \mathcal{J} \cdot \mathcal{K} \neq \{0\}$, where the latter inequality follows from (v). So, since $\mathcal{J} \cap \mathcal{K}$ is a nonzero ideal in $\mathcal{K}$ and $\mathcal{K}$ is simple, we get that $\mathcal{J} \cap \mathcal{K} = \mathcal{K}$ which is obviously in $\mathcal{J}$.

Proof of Proposition 6.1

Let $U_0(t) = e^{itH_0}$, $t \in \mathbb{R}$ where $H_0 = P^2$; then since H_0 is quadratic in P, $\operatorname{Ad} U_0(t)$ induces a symplectic transformation on the resolvent algebra. Thus

$$U_0(t)\,\pi_0(\mathcal{R}(X,\sigma))\,U_0(t)^{-1} \subset \pi_0(\mathcal{R}(X,\sigma))\,, \quad t \in \mathbb{R}\,.$$

To prove that this inclusion still holds if $U_0(t)$ is replaced by $U(t) := e^{itH}$ where $H = P^2 + V(Q)$, we consider the cocycle $\Gamma_V(t) := U(t)\,U_0(t)^{-1}$, $t \in \mathbb{R}$. It will suffice to show that the $\Gamma_V(t) - \mathbb{1}$ are compact for all $V \in C_0(\mathbb{R})$ since then $\Gamma_V(t) \in \pi_0(\mathcal{R}(X,\sigma))$, $t \in \mathbb{R}$ and hence

$$U(t)\pi_0(\mathcal{R}(X,\sigma))U(t)^{-1} = \Gamma_V(t)\,U_0(t)\pi_0(\mathcal{R}(X,\sigma))U_0(t)^{-1}\,\Gamma_V(t)^{-1} \subset \ \pi_0(\mathcal{R}(X,\sigma))\,,$$

by $\Gamma_V(t)^{-1} = \Gamma_V(t)^* \in \pi_0(\mathcal{R}(X,\sigma))$.

We start with the Dyson perturbation series of $\Gamma_V(t)$ given by

$$\Gamma_V(t) = \sum_{n=0}^{\infty} i^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1\, V_{t_1}\, V_{t_2} \cdots V_{t_n}\,,$$

where $V_t := U_0(t)V(Q)U_0(t)^{-1}$, cf. [3] Theorem 3.1.33. The integrals are defined in the strong operator topology. It is an immediate and well–known consequence of this formula that the cocycles $\Gamma_V(t)$, $t \in \mathbb{R}$ are continuous in V. More precisely, putting $\|V\| = \|V(Q)\|$

$$\|\Gamma_{V_1}(t) - \Gamma_{V_2}(t)\| \leq \|V_1 - V_2\| \, \frac{e^{|t|(\|V_1\|+\|V_2\|)} - 1}{\|V_1\|+\|V_2\|} \, .$$

So it suffices to prove compactness of $\Gamma_V(t) - \mathbb{1}$ for a subspace of functions V which are dense in $C_0(\mathbb{R})$, and we will take the space $\{V \in \mathcal{S}(\mathbb{R}) \mid \int dx\, V(x) = 0\}$. For functions in this space, the operators $\int_0^t ds\, V_s$ are Hilbert–Schmidt. To see this, consider their integral kernels in momentum space, which are given by

$$\mathbb{R} \ni (u,v) \mapsto \Big(\int_0^t ds\, V_s \Big)(u,v) = \frac{i}{\sqrt{2\pi}} \, \frac{1 - e^{\, it(u^2 - v^2)}}{(u^2 - v^2)} \, \widetilde{V}(u - v) \, ,$$

where $\widetilde{V}$ denotes the Fourier transform of V. Then the square of the Hilbert–Schmidt norm of the operator is:

$$\Big\| \int_0^t ds\, V_s \Big\|_2^2 = \int du \int dv \, \Big| \Big(\int_0^t ds\, V_s \Big)(u,v) \Big|^2 = |t| \int dw \, \frac{|\widetilde{V}(w)|^2}{2|w|} < \infty \, .$$

The latter bound follows from the fact that $\widetilde{V}$ is a test function which vanishes at the origin. Thus the strong operator continuous functions

$$\mathbb{R}^{n-1} \ni (t_2, \dots t_n) \mapsto \int_0^{t_2} dt_1 \, V_{t_1} \, V_{t_2} \cdots V_{t_n} \, , \ n - 1 \in \mathbb{N}$$

have values in the Hilbert–Schmidt class and their Hilbert–Schmidt norms are bounded by

$$\Big\| \int_0^{t_2} dt_1 \, V_{t_1} \, V_{t_2} \cdots V_{t_n} \Big\|_2^2 \leq |t_2| \int dw \, \frac{|\widetilde{V}(w)|^2}{2|w|} \, \|V\|^{2n-2} \, .$$

In particular, these norms are uniformly bounded on compact subsets of $\mathbb{R}^{n-1}$. But the integral of any strong operator continuous Hilbert–Schmidt valued function with uniformly bounded Hilbert–Schmidt norm is again in the Hilbert–Schmidt class. So we conclude that each term in the above Dyson expansion is a Hilbert–Schmidt operator, except for the first term $\mathbb{1}$ corresponding to $n = 0$. Since the Dyson series converges absolutely in norm, this shows that $\Gamma_V(t) - \mathbb{1}$ is a compact operator for the restricted class of potentials V. The statement for arbitrary $V \in C_0(\mathbb{R})$ then follows from the continuity of $\Gamma_V(t)$ in V.

We mention as an aside that the operators $(U(t)WU(t)^{-1} - U_0(t)WU_0(t)^{-1})$, $t \in \mathbb{R}$ are compact for any bounded operator W as a consequence of the preceding result. Thus if a norm–closed and irreducible subalgebra $\mathcal{W}$ of the algebra of all bounded operators is to be stable under the action of the given family of dynamics it must contain the compact operators. The Weyl algebra, being simple, does not have this feature and therefore does not admit interesting dynamics.

Proof of Proposition 6.2

By the very definition of the resolvent algebra we have $(i\lambda\mathbb{1} - P)^{-1} \in \pi_0(\mathcal{R}(X,\sigma))$, $\lambda \in \mathbb{R}\backslash\{0\}$. As $\pi_0(\mathcal{R}(X,\sigma))$ is a C*–algebra, any continuous, asymptotically vanishing function of P is therefore

an element of $\pi_0(\mathcal{R}(X,\sigma))$ as well, cf. Proposition 5.2. In particular, the resolvent of the free Hamiltonian $H_0 = P^2$ is contained in $\pi_0(\mathcal{R}(X,\sigma))$. Now for $H = P^2 + V(Q)$ we have

$$(i\lambda\mathbb{1} - H)^{-1} = (i\lambda\mathbb{1} - H_0)^{-1} + (i\lambda\mathbb{1} - H)^{-1}\, V(Q)\, (i\lambda\mathbb{1} - H_0)^{-1}\,, \quad \lambda \in \mathbb{R}\backslash\{0\}\,.$$

It follows from standard arguments, cf. [2, 27], that for the given family of potentials V the operators $V(Q)\,(i\lambda\mathbb{1} - H_0)^{-1}$ are compact. Hence the resolvent of H is contained in $\pi_0(\mathcal{R}(X,\sigma))$.

Proof of Proposition 6.3

Let $U(n) := e^{inH}$, $n \in \mathbb{N}$ and let Ω be any normalized vector in the underlying Hilbert space. We define a corresponding sequence of states ω_n on $\mathcal{R}(X,\sigma)$, putting

$$\omega_n(R) := (\Omega, U(n)\pi_0(R)U(n)^{-1}\Omega)\,, \quad R \in \mathcal{R}(X,\sigma)\,.$$

As the Hamiltonian $H = P^2 - Q^2$ is quadratic, one obtains by a routine calculation, $f_\pm \in \mathbb{R}$,

$$U(n)(i\lambda\mathbb{1} - f_+(P+Q) - f_-(P-Q))^{-1}U(n)^{-1} = (i\lambda\mathbb{1} - e^{2n}f_+(P+Q) - e^{-2n}f_-(P-Q))^{-1}\,.$$

Hence, by the same reasoning as in the proof of Proposition 4.8, one finds that

$$\operatorname*{s-lim}_{n\to\infty} U(n)(i\lambda\mathbb{1} - f_+(P+Q) - f_-(P-Q))^{-1}U(n)^{-1} = \begin{cases} 0 & \text{if } f_+ \neq 0 \\ \frac{1}{i\lambda}\,\mathbb{1} & \text{if } f_+ = 0 \end{cases}$$

It follows that the sequence of states ω_n, $n \in \mathbb{N}$ converges pointwise on $\mathcal{R}(X,\sigma)$ and that its limit ω_∞ induces a one–dimensional representation of $\mathcal{R}(X,\sigma)$.
Assume now that there is some pseudo–resolvent $R_\lambda \in \mathcal{R}(X,\sigma)$ such that $\pi_0(R_\lambda) = (i\lambda\mathbb{1} - H)^{-1}$, $\lambda \in \mathbb{R}\backslash 0$. By the preceding result and the resolvent equation for R_λ we then have $\omega_\infty(R_\lambda) = (i\lambda - \nu)^{-1}$ for some $\nu \in \mathbb{R} \cup \{\infty\}$. On the other hand it follows from the definition of the states ω_n that $\omega_n(R_\lambda) = (\Omega, (i\lambda\mathbb{1} - H)^{-1}\,\Omega)$, $n \in \mathbb{N}$. Hence we conclude that

$$(\Omega, (i\lambda\mathbb{1} - H)^{-1}\,\Omega) = (i\lambda - \nu)^{-1}\,, \quad \lambda \in \mathbb{R}\backslash 0\,.$$

But this is impossible since H has continuous spectrum as $-Q^2$ is a repulsive potential.

Proof of Proposition 7.1

(i) In this somewhat lengthy proof we will state intermediate results in italics if they are of interest in their own right. We begin by gathering notation and elementary facts. For $\Lambda \subset \mathbb{Z} \ni l$, let

$$H_\Lambda^{(0)} := \sum_{l\in\Lambda}(P_l^2 + Q_l^2)\,, \qquad U_\Lambda^{(0)}(t) := e^{itH_\Lambda^{(0)}}\,,\ t \in \mathbb{R}\,,$$
$$K_l^{(0)} := \frac{1}{2}((P_l - P_{l+1})^2 + (Q_l - Q_{l+1})^2)\,, \qquad V_l^{(0)}(t) := e^{itK_l^{(0)}}\,,\ t \in \mathbb{R}\,.$$

Since $U_\Lambda^{(0)}(t) \in \pi_0(\mathcal{R}(X_\Lambda,\sigma))''$ and $\mathrm{Ad}\,U_\Lambda^{(0)}(t)$ induce symplectic transformations on $\mathcal{R}(X_\Lambda,\sigma)$, we have for any $\Lambda_0 \subset \Lambda$ that

$$U_\Lambda^{(0)}(t)\,R_0\,U_\Lambda^{(0)}(t)^{-1} = U_{\Lambda_0}^{(0)}(t)\,R_0\,U_{\Lambda_0}^{(0)}(t)^{-1} \in \pi_0(\mathcal{R}(X_{\Lambda_0},\sigma))\,, \qquad R_0 \in \pi_0(\mathcal{R}(X_{\Lambda_0},\sigma))\,.$$

Furthermore, by

$$2(P_l^2 + Q_l^2 + P_{l+1}^2 + Q_{l+1}^2) = ((P_l - P_{l+1})^2 + (Q_l - Q_{l+1})^2) + ((P_l + P_{l+1})^2 + (Q_l + Q_{l+1})^2)$$

we have for $l, l+1 \in \Lambda$ that

$$U_\Lambda^{(0)}(t)\, R_l\, U_\Lambda^{(0)}(t)^{-1} = V_l^{(0)}(t)\, R_l\, V_l^{(0)}(t)^{-1} \in \pi_0(\mathcal{R}(Y_l,\sigma))\,, \quad R_l \in \pi_0(\mathcal{R}(Y_l,\sigma))\,, \tag{24}$$

where $Y_l := \mathrm{Span}\{p_l - p_{l+1}, q_l - q_{l+1}\}$.

After these preparations we start our proof of (i). We intend to show that the cocycles $\Gamma_\Lambda(t) := U_\Lambda(t)\, U_\Lambda^{(0)}(t)^{-1}$, $t \in \mathbb{R}$ are in $\pi_0(\mathcal{R}(X_\Lambda,\sigma))$. As in the proof of Proposition 6.1 consider the Dyson expansion of $\Gamma_\Lambda(t)$ which now takes the form

$$\begin{aligned} \Gamma_\Lambda(t) &= \sum_{n=0}^\infty i^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1\, V_{\Lambda,t_1}\, V_{\Lambda,t_2} \cdots V_{\Lambda,t_n}\,, \\ V_{\Lambda,t} &:= \sum_{l,l+1\in\Lambda} V_{l,t}\,, \quad V_{l,t} := U_\Lambda^{(0)}(t)\, V(Q_l - Q_{l+1})\, U_\Lambda^{(0)}(t)^{-1} = V_l^{(0)}(t)\, V(Q_l - Q_{l+1})\, V_l^{(0)}(t)^{-1}\,, \end{aligned}$$

where the latter equality follows from equation (24) via $V(Q_l - Q_{l+1}) \in \pi_0(\mathcal{R}(Y_l,\sigma))$. We will show that each summand in the Dyson expansion is in $\pi_0(\mathcal{R}(X_\Lambda,\sigma))$. Consider the first non–trivial term. For its building blocks we have:

The functions $\mathbb{R} \ni t \mapsto \int_0^t ds\, V_{l,s}$ are Hölder continuous in the norm topology and their values are in $\pi_0(\mathcal{R}(Y_l,\sigma))$, $l \in \mathbb{Z}$.

The Hölder continuity is obvious by $\| \int_0^{t_1} ds\, V_{l,s} - \int_0^{t_2} ds\, V_{l,s}\| \le |t_1 - t_2|\|V\|$. Since $V_{l,s} \in \pi_0(\mathcal{R}(Y_l,\sigma))$ and the integral is defined in the strong operator topology, it is also clear that $\int_0^t ds\, V_{l,s} \in \pi_0(\mathcal{R}(Y_l,\sigma))''$. But $\pi_0(\mathcal{R}(Y_l,\sigma))''$ is a factor, so for the second part of the statement it suffices to prove that $\int_0^t ds\, V_{l,s} \restriction \mathcal{H}_0(Y_l) \in \pi_0(\mathcal{R}(Y_l,\sigma)) \restriction \mathcal{H}_0(Y_l)$, where $\mathcal{H}_0(Y_l) = \overline{\pi_0(\mathcal{R}(Y_l,\sigma))\Omega_0}$. This will be done by proving that $\int_0^t ds\, V_{l,s} \restriction \mathcal{H}_0(Y_l)$ is compact, using the fact that $\pi_0 \restriction \mathcal{R}(Y_l,\sigma)$ on $\mathcal{H}_0(Y_l)$ is equivalent to the Schrödinger representation of $\mathcal{R}(Y_l,\sigma)$.

Define the canonical operators $Q = \frac{1}{\sqrt{2}}(Q_l - Q_{l+1}) \restriction \mathcal{H}_0(Y_l)$ and $P = \frac{1}{\sqrt{2}}(P_l - P_{l+1}) \restriction \mathcal{H}_0(Y_l)$, then

$$C := \int_0^t ds\, V_{l,s} \restriction \mathcal{H}_0(Y_l) = \int_0^t ds\, e^{is(P^2+Q^2)}\, V(\sqrt{2}\,Q)\, e^{-is(P^2+Q^2)}\,.$$

Let $\Phi_n \in \mathcal{H}_0(Y_l)$ be the orthonormal basis of eigenvectors of $P^2 + Q^2$ corresponding to the eigenvalues $2n+1$, $n = 0, 1, 2, \ldots$, then for $n \ne m$ we have

$$C_{mn} := \Big(\Phi_m, \int_0^t ds\, e^{is(P^2+Q^2)}\, V(\sqrt{2}\,Q)\, e^{-is(P^2+Q^2)}\Phi_n\Big) = \frac{e^{2it(m-n)} - 1}{2i(m-n)}\,(\Phi_m, V(\sqrt{2}\,Q)\Phi_n)\,,$$

where for $m = n$ the fraction has to be replaced by t. We need to estimate the matrix elements of the potential on the rhs, and will first consider potentials $V \in C_0(\mathbb{R})$ with compact support. Then

$$|(\Phi_m, V(\sqrt{2}\,Q)\Phi_n)| \le \|V(\sqrt{2}Q)\, e^{Q^2}\|\, \|e^{-\frac{1}{2}Q^2}\Phi_m\|\, \|e^{-\frac{1}{2}Q^2}\Phi_n\|\,.$$

From the standard representation of Φ_n in configuration space by Hermite functions,

$$x \mapsto \Phi_n(x) = (-1)^n \, (2^n n!)^{-1/2} \pi^{-1/4} \, e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2} \, ,$$

one gets by proceeding to Fourier transforms and making use of Parseval's Theorem

$$\|e^{-\frac{1}{2}Q^2} \Phi_n\|^2 = \frac{1}{\sqrt{2}} \, \frac{(2n)!}{2^{2n}(n!)^2} \, .$$

Then the estimate (cf. [24])

$$\sqrt{2\pi} \, n^{n+1/2} e^{-n+(12n+1)^{-1}} < n! < \sqrt{2\pi} \, n^{n+1/2} e^{-n+(12n)^{-1}} \quad \text{for} \quad n \geq 1$$

implies that

$$\|e^{-\frac{1}{2}Q^2} \Phi_n\|^2 \leq n^{-1/2} \quad \text{for} \quad n \geq 1 \, ,$$

and for $n = 0$ one has $\|e^{-\frac{1}{2}Q^2} \Phi_0\|^2 \leq 1$. Thus one obtains

$$|C_{m\,n}| \leq \begin{cases} K/(m^{1/4} n^{1/4} \, |m-n|) & \text{if } m \neq n \text{ and } m, \, n > 0 \\ K/n^{5/4} & \text{if } m = 0, n > 0 \\ K/m^{5/4} & \text{if } n = 0, m > 0 \\ |t| K/n^{1/2} & \text{if } m = n > 0 \\ |t| K & \text{if } m = n = 0 \end{cases} \quad .$$

where $K := \|V(\sqrt{2}\,Q) e^{Q^2}\|$.

These bounds will enable us to show that C is a compact operator. Note first that the operators $C_N := C \cdot P_N$, $N \in \mathbb{N}$ where P_N is the projection onto $\mathrm{Span}\{\Phi_1, \dots, \Phi_N\}$, are compact as C is bounded and P_N is finite rank. Hence to prove that C is a compact operator, it suffices to show that $\|C - C_N\| \to 0$ as $N \to \infty$. This can be accomplished by Schur's test according to which the norm of an operator D satisfies the bound $\|D\| \leq \sqrt{ab}$ if there exist $a, b \in \mathbb{R}_+$ such that $\sup_n \sum_m |D_{m\,n}| < a$ and $\sup_m \sum_n |D_{m\,n}| < b$ where $D_{m\,n}$ denotes its matrix element w.r.t. a given orthonormal basis of the Hilbert space on which it acts. Using the preceding bounds on the matrix elements $C_{m\,n}$ one can show by a routine computation that $\sup_{n \geq N} \sum_m |C_{m\,n}| \to 0$ as $N \to \infty$ and similarly $\sum_{n \geq N} |C_{m\,n}| \to 0$ uniformly in m as $N \to \infty$. It follows that $\|C - C_N\| \to 0$ for $N \to \infty$. Thus C is compact for the restricted class of potentials V and this result extends to arbitrary potentials by norm continuity of C in $V \in C_0(\mathbb{R})$.

Since $Y_l \subset X_\Lambda$ for $l, l+1 \in \Lambda$ we thus have shown that $\int_0^t ds \, V_{\Lambda,s} = \int_0^t ds \sum\limits_{l,l+1 \in \Lambda} V_{l,s} \in \pi_0(\mathcal{R}(X_\Lambda, \sigma))$. For the proof that the remaining terms in the Dyson expansion are also contained in $\pi_0(\mathcal{R}(X_\Lambda, \sigma))$ we make use of the following fact.

For a C^*–algebra $\mathcal{C}$ on a Hilbert space $\mathcal{H}$, let $\mathbb{R} \ni s \mapsto A_s \in \mathcal{C}$ be Hölder continuous on compact subsets of $\mathbb{R}$ in the norm topology, let $\mathbb{R} \ni s \mapsto B_s \in \mathcal{B}(\mathcal{H})$ be continuous in the strong operator topology, and let $\int_0^t ds \, B_s \in \mathcal{C}$, $t \in \mathbb{R}$. Then $\mathbb{R} \ni t \mapsto C_t := \int_0^t ds \, B_s A_s$ is Hölder continuous in the norm topology on compact subsets of $\mathbb{R}$ and $C_t \in \mathcal{C}$, $t \in \mathbb{R}$.

We prove this. By the assumptions, the operators

$$C_t^{(n)} := \sum_{m=0}^{n-1} \int_{tm/n}^{t(m+1)/n} ds \, B_s \, A_{t(m+1/2)/n}$$

are in $\mathcal{C}$ for any $n \in \mathbb{N}$. But

$$\|C_t - C_t^{(n)}\| \leq \frac{|t|}{n} \sup_{0 \leq s \leq t} \|B_s\| \sum_{m=0}^{n-1} \sup_{mt/n \leq s \leq t(m+1)/n} \|A_s - A_{t(m+1/2)/n}\| \leq c\, \frac{|t|^{1+h}}{n^h} \,,$$

where in the second inequality we used the assumption that $\|A_{s_1} - A_{s_2}\| \leq c' \, |s_1 - s_2|^h$ on compact subsets of $\mathbb{R}$, and c is a constant depending on B_s, c' and h. Thus C_t can be approximated in norm by elements of $\mathcal{C}$ and hence is in $\mathcal{C}$ as well. For the Hölder–continuity, note that

$$\|C_{t_1} - C_{t_2}\| \leq |t_1 - t_2| \sup_{t_1 \leq s \leq t_2} \|B_s A_s\| \leq c\, |t_1 - t_2|$$

on compact subsets of $\mathbb{R}$.

We now use the preceding two results to prove by induction that all terms in the Dyson expansion are in $\pi_0(\mathcal{R}(X_\Lambda, \sigma))$. For $n = 1$ we have already shown that $t \mapsto S_t^{(1)} := \int_0^t ds\, V_{\Lambda,s}$ is Hölder continuous in norm (on compact subsets of $\mathbb{R}$) with values in $\pi_0(\mathcal{R}(X_\Lambda, \sigma))$. Assume that

$$t \mapsto S_t^{(n)} := \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1\, V_{\Lambda,t_1}\, V_{\Lambda,t_2} \cdots V_{\Lambda,t_n}$$

has these properties too. For the inductive step, note that the Hölder continuity of $S_t^{(n+1)}$ follows from the estimate

$$\begin{aligned} \|S_t^{(n+1)} - S_{t'}^{(n+1)}\| &\leq \Big| \int_{t'}^t dt_{n+1} \Big| \int_0^{t_{n+1}} dt_n \cdots \Big| \int_0^{t_2} dt_1 \Big| \cdots \Big|\Big| \sup_{t_1, \dots, t_n} \|V_{\Lambda,t_1}\, V_{\Lambda,t_2} \cdots V_{\Lambda,t_{n+1}}\| \\ &\leq \frac{|\epsilon(t)\, |t|^{n+1} - \epsilon(t')\, |t'|^{n+1}|}{(n+1)\,!}\; |\Lambda|^{n+1} \|V\|^{n+1} \,, \end{aligned}$$

where $|\Lambda|$ is the number of points in Λ and ϵ the sign–function. But $S_t^{(n+1)} = \int_0^t dt_{n+1} V_{\Lambda,t_{n+1}} S_{t_{n+1}}^{(n)}$, so it follows from the induction basis and hypothesis by an application of the preceding general result that $S_t^{(n+1)} \in \pi_0(\mathcal{R}(X_\Lambda, \sigma))$, completing the induction. Since the Dyson series converges absolutely in norm we conclude that also $\Gamma_\Lambda(t) \in \pi_0(\mathcal{R}(X_\Lambda, \sigma))$, $t \in \mathbb{R}$. Since the adjoint action of $U_\Lambda^{(0)}(t)$ leaves $\pi_0(\mathcal{R}(X_\Lambda, \sigma))$ invariant it is then clear that

$$U_\Lambda(t) \pi_0(\mathcal{R}(X_\Lambda, \sigma)) U_\Lambda(t)^{-1} = \Gamma_\Lambda(t) U_\Lambda^{(0)}(t) \pi_0(\mathcal{R}(X_\Lambda, \sigma)) U_\Lambda^{(0)}(t)^{-1} \Gamma_\Lambda(t)^{-1} \subset \pi_0(\mathcal{R}(X_\Lambda, \sigma)) \,.$$

(ii) We use a standard argument from the theory of spin systems [3] for this part. By the net structure of $\mathcal{R}(X, \sigma)$ it suffices to prove the claim for the sets $\Lambda_0 := \{l \in \mathbb{N} \mid |l| \leq n_0\} \subset \mathbb{Z}$ and henceforth we fix such a Λ_0, hence an n_0, and let $R_0 \in \pi_0(\mathcal{R}(X_{\Lambda_0}, \sigma))$.

To prove the claimed convergence, we start with the Dyson perturbation series for the adjoint action of the cocycle $\Gamma_\Lambda(t) := U_\Lambda(t)\, U_\Lambda^{(0)}(t)^{-1}$:

$$\Gamma_\Lambda(t) R_0 \Gamma_\Lambda(t)^{-1} = R_0 + \sum_{n=1}^\infty i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, [V_{\Lambda,t_n}, [V_{\Lambda,t_{n-1}}, \cdots, [V_{\Lambda,t_1}, R_0] \cdots]] \,. \quad (25)$$

Since $V_{l,t} \in \pi_0(\mathcal{R}(S_l, \sigma))$, $t \in \mathbb{R}$ it commutes with $\pi_0(\mathcal{R}(X_{\Lambda_n}, \sigma))$ if $l > n_0 + n$ or $l + 1 < -n_0 - n$ where $\Lambda_n := \{l \in \mathbb{N} \mid |l| \leq n_0 + n\}$. So by $R_0 \in \pi_0(\mathcal{R}(X_{\Lambda_0}, \sigma))$ we have

$$[V_{\Lambda,t_n}, [V_{\Lambda,t_{n-1}}, \cdots, [V_{\Lambda,t_1}, R_0] \cdots]] = [V_{\Lambda\cap\Lambda_n,t_n}, [V_{\Lambda\cap\Lambda_{n-1},t_{n-1}}, \cdots, [V_{\Lambda\cap\Lambda_1,t_1}, R_0] \cdots]]$$

and hence

$$\begin{aligned}
&\|[V_{\Lambda,t_n},[V_{\Lambda,t_{n-1}},\cdots,[V_{\Lambda,t_1},R_0]\cdots]]\| \\
&\leq 2^n \|\sum_{l\in\Lambda\cap\Lambda_n} V_{l,0}\|\cdots\|\sum_{l'\in\Lambda\cap\Lambda_1} V_{l',0}\|\,\|R_0\| \leq 4^n(n_0+1)\cdots(n_0+n)\|V\|^n\|R_0\|\,.
\end{aligned}$$

Moreover, if Λ, Λ' are regions which both contain Λ_N for some $N \in \mathbb{N}$, then the first N terms in the respective Dyson series coincide and so

$$\|\Gamma_\Lambda(t)R_0\Gamma_\Lambda(t)^{-1} - \Gamma_{\Lambda'}(t)R_0\Gamma_{\Lambda'}(t)^{-1}\| \leq \sum_{n=N+1}^{\infty} \frac{|t|^n}{n\,!} 4^n(n_0+1)\cdots(n_0+n)\|V\|^n\|R_0\|\,.$$

The upper bound exists if $|t| < \frac{1}{4\|V\|}$ and it tends to 0 as $N \to \infty$. Thus for $R_0 \in \pi_0(\mathcal{R}(X_{\Lambda_0},\sigma))$ and sufficiently small $|t|$ the nets $\{\Gamma_\Lambda(t)R_0\Gamma_\Lambda(t)^{-1}\}_{\Lambda\subset\mathbb{Z}}$ converge (uniformly) as $\Lambda \nearrow \mathbb{Z}$. Since $U_\Lambda^{(0)}(t)R_0U_\Lambda^{(0)}(t)^{-1} = U_{\Lambda_0}^{(0)}(t)R_0U_{\Lambda_0}^{(0)}(t)^{-1} \in \pi_0(\mathcal{R}(X_{\Lambda_0}))$ by the remarks at the beginning of this proof and hence

$$U_\Lambda(t)R_0U_\Lambda(t)^{-1} = \Gamma_\Lambda(t)\, U_{\Lambda_0}^{(0)}(t)R_0U_{\Lambda_0}^{(0)}(t)^{-1}\,\Gamma_\Lambda(t)^{-1}$$

we conclude that the nets $\{U_\Lambda(t)R_0U_\Lambda(t)^{-1}\}_{\Lambda\subset\mathbb{Z}}$ also converge in norm as $\Lambda \nearrow \mathbb{Z}$. However $\bigcup_{\Lambda_0\subset\mathbb{Z}} \pi_0(\mathcal{R}(X_{\Lambda_0},\sigma))$ is norm dense in $\pi_0(\mathcal{R}(X,\sigma))$ and the adjoint action of unitary operators is norm continuous, so the existence of the norm limits

$$\beta_t(R) := \text{n-}\lim_{\Lambda\nearrow\mathbb{Z}} U_\Lambda(t)\pi_0(R)U_\Lambda(t)^{-1}\,, \quad R \in \mathcal{R}(X,\sigma) \tag{26}$$

follows if $|t| < \frac{1}{4\|V\|}$. Moreover, $\beta_t(\pi_0(\mathcal{R}(X,\sigma)) \subset \pi_0(\mathcal{R}(X,\sigma))$, $|t| < \frac{1}{4\|V\|}$ by part (i) above. By the group property $U_\Lambda(s+t) = U_\Lambda(s)U_\Lambda(t)$ one finds by repeated application of the preceding two results that these statements hold for arbitrary $t \in \mathbb{R}$ which proves this part.
(iii) Recalling that the representation π_0 of $\mathcal{R}(X,\sigma)$ is faithful, one can define an automorphic action of the group $\mathbb{R}$ on $\mathcal{R}(X,\sigma)$ induced by the set of Hamiltonians $\{H_\Lambda\}_{\Lambda\in\mathbb{Z}}$, putting for $t \in \mathbb{R}$

$$\alpha_t(R) := \pi_0^{-1}(\beta_t(\pi_0(R)))\,, \quad R \in \mathcal{R}(X,\sigma)\,.$$

This completes the proof of the proposition.

Proof of Proposition 7.2

Let $\Lambda_0 = \{l \in \mathbb{N} \mid |l| \leq n_0\} \subset \mathbb{Z}$ and let $R_0 \in \pi_0(\mathcal{R}(X_{\Lambda_0},\sigma))$. In the preceding proof of Proposition 7.1 we established the existence of the norm limits

$$\gamma_t(R_0) := \text{n-}\lim_{\Lambda\nearrow\mathbb{Z}} \Gamma_\Lambda(t)R_0\Gamma_\Lambda(t)^{-1}\,, \quad R_0 \in \pi_0(\mathcal{R}(X_{\Lambda_0},\sigma))\,.$$

From the expansion (25) and the remarks subsequent to it we also obtain for $|t|, |t'| < \frac{1}{4\|V\|}$ the uniform bound for $\Lambda \subset \mathbb{Z}$

$$\|\Gamma_\Lambda(t)R_0\Gamma_\Lambda(t)^{-1} - \Gamma_\Lambda(t')R_0\Gamma_\Lambda(t')^{-1}\| \leq \sum_{n=1}^{\infty} \frac{|\epsilon(t)\,|t|^n - \epsilon(t')\,|t'|^n|}{n\,!} 4^n(n_0+1)\cdots(n_0+n)\|V\|^n\|R_0\|\,.$$

Combining these results it follows that γ_t acts norm continuously on the elements of $\pi_0(\mathcal{R}(X_{\Lambda_0},\sigma))$ for $|t| < \frac{1}{4\|V\|}$, and this continuity property extends to all of $\pi_0(\mathcal{R}(X,\sigma))$ since $\bigcup_{\Lambda_0\subset\mathbb{Z}} \pi_0(\mathcal{R}(X_{\Lambda_0},\sigma))$ is norm dense in $\pi_0(\mathcal{R}(X,\sigma))$. Next, putting

$$\beta_t^{(0)}(R) := \text{n--}\lim_{\Lambda\nearrow\mathbb{Z}} U_\Lambda^{(0)}(t)\, R\, U_\Lambda^{(0)}(t)^{-1}\,, \quad R \in \pi_0(\mathcal{R}(X,\sigma))$$

we infer from the remarks at the beginning of the proof of Proposition 7.1 that

$$\beta_t^{(0)} \restriction \pi_0(\mathcal{R}(X_{\Lambda_0},\sigma)) = (\operatorname{Ad} U_{\Lambda_0}^{(0)}(t)) \restriction \pi_0(\mathcal{R}(X_{\Lambda_0},\sigma)) \subset \pi_0(\mathcal{R}(X_{\Lambda_0},\sigma))\,.$$

It follows that $\beta_t^{(0)}$, $t \in \mathbb{R}$ acts norm continuously on the (up to multiplicity) compact operators in $\pi_0(\mathcal{K}_{\Lambda_0}) \subset \pi_0(\mathcal{R}(X_{\Lambda_0},\sigma))$, $\Lambda_0 \subset \mathbb{Z}$ and hence also on their inductive limit $\pi_0(\mathcal{K})$.
Now for the full time evolution (26) we have $\beta_t = \gamma_t \circ \beta_t^{(0)}$, $t \in \mathbb{R}$. So summarizing the preceding results we conclude that β_t acts pointwise norm continuously on $\pi_0(\mathcal{K})$ for small t. In view of $\|\beta_t(\pi_0(K)) - \beta_{t'}(\pi_0(K))\| = \|\beta_{t-t'}(\pi_0(K)) - \pi_0(K)\|$ this statement extends to arbitrary $t \in \mathbb{R}$ and it is then clear that α_t acts pointwise norm continuously on $\mathcal{K}$, $t \in \mathbb{R}$, proving the statement.

Proof of Lemma 7.3

In addition to the operators H_n, $\widetilde{H}_n$, $n \in \mathbb{N}$ introduced in the main text we will consider here also the operators $H_{n\backslash m} := H_{\Lambda_{n\backslash m}}$ corresponding to the sets $\Lambda_{n\backslash m} := \{l \in \mathbb{Z} \,|\, m < |l| \le n\}$ and their renormalized versions $\widetilde{H}_{n\backslash m} = H_{n\backslash m} - E_{n\backslash m}\mathbb{1}$, where $E_{n\backslash m}$ is the smallest eigenvalue of $H_{\Lambda_{n\backslash m}}$. Since $\widetilde{H}_m$ and $\widetilde{H}_{n\backslash m}$ commute and the potential V is bounded, the domains of these operators are related by $\mathcal{D}(\widetilde{H}_n) = \mathcal{D}(\widetilde{H}_m) \cap \mathcal{D}(\widetilde{H}_{n\backslash m})$, and on the latter domain we have the operator equality

$$\widetilde{H}_n = \widetilde{H}_m + \widetilde{H}_{n\backslash m} + V(Q_{-m-1} - Q_{-m}) + V(Q_m - Q_{m+1}) + (E_m + E_{n\backslash m} - E_n)\mathbb{1}\,.$$

Let Ω be a normalised joint eigenvector for $\widetilde{H}_m$ and $\widetilde{H}_{n\backslash m}$ for the eigenvalue zero, then $(\Omega, \widetilde{H}_n\Omega) \ge 0$ implies via the last equation that:

$$(E_m + E_{n\backslash m} - E_n) \ge -2\,\|V\|\,.$$

Consequently $\widetilde{H}_n \ge \widetilde{H}_m - 4\,\|V\|\mathbb{1}$ and hence for their resolvents we have:

$$((\mu + 4\|v\|)\mathbb{1} + \widetilde{H}_n)^{-1} \le (\mu\mathbb{1} + \widetilde{H}_m)^{-1} \le \mu^{-1}\mathbb{1} \quad \text{for all}\, \mu > 0;\ m < n,\ m, n \in \mathbb{N}\,.$$

Recalling that $\widetilde{H}_m$ is affiliated with $\mathcal{K}_{\Lambda_m} \subset \mathcal{R}(\Lambda_m,\sigma)$, let $\widetilde{R}_m(\mu) \in \mathcal{K}_{\Lambda_m}$, $\mu > 0$ be the corresponding pseudo resolvent, *i.e.* $\pi_0(\widetilde{R}_m(\mu)) = (\mu\mathbb{1} + \widetilde{H}_m)^{-1}$, $m \in \mathbb{N}$. Since ω_∞ is a w*–limit point of $\{\omega_n\}_{n\in\mathbb{N}}$ there is a subsequence $\{\omega_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\omega_\infty(\widetilde{R}_m(\mu)) = \lim_{k\to\infty} (\Omega_{n_k}, \pi_0(\widetilde{R}_m(\mu))\Omega_{n_k}) = \lim_{k\to\infty} (\Omega_{n_k}, (\mu\mathbb{1} + \widetilde{H}_m)^{-1}\Omega_{n_k})$$

for fixed m and μ. The preceding operator inequality and the fact that $\widetilde{H}_{n_k}\Omega_{n_k} = 0$, $k \in \mathbb{N}$ imply

$$(\mu + 4\|V\|)^{-1} = \lim_{k\to\infty} (\Omega_{n_k}, ((\mu + 4\|V\|)\mathbb{1} + \widetilde{H}_{n_k})^{-1}\Omega_{n_k}) \le \lim_{k\to\infty} (\Omega_{n_k}, (\mu\mathbb{1} + \widetilde{H}_m)^{-1}\Omega_{n_k}) \le \mu^{-1}\,.$$

Hence $(\mu+4\|V\|)^{-1} \le \omega_\infty(\widetilde{R}_m(\mu)) \le \mu^{-1}$, $\mu > 0$ and so $\lim_{\mu\to\infty} \omega_\infty(\mu\,\widetilde{R}_m(\mu)) = 1$. As $\|\mu\,\widetilde{R}_m(\mu)\| \le 1$ and $\widetilde{R}_m(\mu) \in \mathcal{K}_{\Lambda_m}$ it follows that $\|\omega_\infty \restriction \mathcal{K}_{\Lambda_m}\| = 1$, $m \in \mathbb{N}$. Thus the GNS representation of ω_∞ is nondegenerate on $\mathcal{K}_{\Lambda_m}$ and hence by Theorem 5.4 (iii), ω_∞ is regular on $\mathcal{R}(X_{\Lambda_m},\sigma)$. This holds for all $m \in \mathbb{N}$ hence ω_∞ is regular.

Proof of Proposition 8.1

(i) First, observe that if $\omega \in \mathfrak{S}_D$ then $(i\lambda R(\lambda, f) - \mathbb{1}) \in \mathcal{N}_\omega \subset \operatorname{Ker}\omega$ for $f \in C$ so putting $\lambda = 1$ we get that $\omega(R(1, f)) = -i$. Next, assume that $\omega(R(1, f)) = -i$ for all $f \in C$. Then, using $\lambda R(\lambda, f) = R(1, \frac{1}{\lambda} f)$ we get that $(i\lambda R(\lambda, f) - \mathbb{1}) \in \operatorname{Ker}\omega$ for $\lambda \in \mathbb{R}$. Now by Eqs. (2) and (4)

$$(i\lambda R(\lambda, f) - \mathbb{1})^*(i\lambda R(\lambda, f) - \mathbb{1}) = \tfrac{i\lambda}{2}\,(R(-\lambda, f) - R(\lambda, f)) + \ \mathbb{1}$$

and hence $\omega\left((i\lambda R(\lambda, f) - \mathbb{1})^*(i\lambda R(\lambda, f) - \mathbb{1})\right) = 0$ for $f \in C$, *i.e.* $\omega \in \mathfrak{S}_D$. Note that as $(i\lambda R(\lambda, f) - \mathbb{1})$ is a normal operator one also has $\omega\left((i\lambda R(\lambda, f) - \mathbb{1})(i\lambda R(\lambda, f) - \mathbb{1})^*\right) = 0$.

(ii) It suffices to prove that if $\sigma(g, C) \neq 0$ for $g \in X$, then $\omega(R(\mu, g)) = 0$ for all $\mu \in \mathbb{R}\backslash 0$, since then $\pi_\omega(R(\lambda, g)) = 0$ by Theorem 4.1(iv). Let $f \in C$ and $\omega \in \mathfrak{S}_D$ then by $(i\lambda R(\lambda, f) - \mathbb{1}) \in \mathcal{N}_\omega \cap \mathcal{N}_\omega^*$ and Eq. (5) we have

$$0 = \omega\left([(i\lambda R(\lambda, f) - 1),\, R(\mu, g)]\right) = -\omega(\lambda\,\sigma(f, g)\,R(\lambda, f)\,R(\mu, g)^2 R(\lambda, f)) = \tfrac{1}{\lambda}\,\sigma(f, g)\,\omega(R(\mu, g)^2)$$

so $\omega(R(\mu, g)^2) = 0$. Now by Eq. (4) and the continuity properties of the resolvents we have

$$i\tfrac{d}{d\mu} R(\mu, g) = i\,\text{n--}\lim_{\lambda \to \mu}\,(\mu - \lambda)^{-1}\,(R(\mu, g) - R(\lambda, g)) = \text{n--}\lim_{\lambda \to \mu}\, R(\mu, g) R(\lambda, g) = R(\mu, g)^2\,.$$

Hence $i\,\frac{d}{d\mu}\omega(R(\mu, g)) = \omega(R(\mu, g)^2) = 0$ and consequently $\mu \mapsto \omega(R(\mu, g)) = \text{const.}$ But $|\omega(R(\mu, g))| \leq \|R(\mu, g)\| \leq \frac{1}{|\mu|}$ and thus $\omega(R(\mu, g)) = 0$.

(iii) From part (ii) note that if $\sigma(C, C) \neq 0$, then for $f, g \in C$ with $\sigma(f, g) \neq 0$ we have that $\omega(i\lambda R(\lambda, f) - \mathbb{1}) = 0$ for all λ implies that $\omega(R(\mu, g)) = 0$, which contradicts with the requirement that $\omega(i\lambda R(\mu, g) - \mathbb{1}) = 0$. Thus $\sigma(C, C) \neq 0$ implies that $\mathfrak{S}_D = \emptyset$.

For the converse let $\sigma(C, C) = 0$, then $\mathcal{R}(C) := \mathrm{C}^*\{R(\lambda, f) \mid \lambda \in \mathbb{R}\backslash 0,\ f \in C\} + \mathbb{C}\mathbb{1}$ is a unital commutative C*–algebra. It is easily checked that the linear map from all polynomials in the resolvents in $\mathcal{R}(C)$ to the complex numbers given by

$$\omega(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n)) := \prod_{k=1}^{n} (1/i\lambda_k)\,, \quad \lambda_1 \ldots \lambda_n \in \mathbb{R}\backslash 0\,,\ f_1 \ldots f_n \in C$$

is a *–homomorphism. Hence it extends to a character on $\mathcal{R}(C)$, *i.e.* a pure state, and then, by the Hahn–Banach theorem, to a state on $\mathcal{R}(X, \sigma)$. By its very construction, $\omega \in \mathfrak{S}_D$.

Proof of Proposition 8.2

The proofs of the facts listed here already appeared elsewhere [16], but we recall them here for completeness.

(i) Recall that $\mathcal{N} := [\mathcal{R}(X, \sigma)\mathcal{C}] = \cap\,\{\mathcal{N}_\omega \mid \omega \in \mathfrak{S}_D\}$ so since $\mathcal{N}_\omega \subset \operatorname{Ker}\omega$ for all ω it is clear that $\mathcal{D} = \mathcal{N} \cap \mathcal{N}^* \subset \bigcap\{\operatorname{Ker}\omega \mid \omega \in \mathfrak{S}_D\}$. Since $\mathcal{D}$ is the intersection of a closed left ideal with its adjoint, we see that $\mathcal{D}$ is a C*–algebra. Next, let $\mathcal{A}$ be any C*–algebra in $\bigcap\{\operatorname{Ker}\omega \mid \omega \in \mathfrak{S}_D\}$. Since $\mathcal{A}$ is a C*–algebra, $A \in \mathcal{A}$ implies that $AA^* \in \mathcal{A} \ni A^*A$ hence $A \in \mathcal{N}_\omega \cap \mathcal{N}_\omega^*$ for all $\omega \in \mathfrak{S}_D$. Thus $\mathcal{A} \subseteq \bigcap\{\mathcal{N}_\omega \cap \mathcal{N}_\omega^* \mid \omega \in \mathfrak{S}_D\} = \mathcal{D}$ and so $\mathcal{D}$ is the maximal C*–algebra annihilated by all Dirac states.

To see that $\mathcal{D}$ is hereditary, use Theorem 3.2.1 in [21] and the fact that $\mathcal{N} = [\mathcal{R}(X,\sigma)\mathcal{C}]$ is a closed left ideal of $\mathcal{R}(X,\sigma)$.

(ii) Since $\mathcal{D}$ is a two–sided ideal for the relative multiplier algebra $M_{\mathcal{R}(X,\sigma)}(\mathcal{D})$ of $\mathcal{D}$ in $\mathcal{R}(X,\sigma)$ it is obvious that $M_{\mathcal{R}(X,\sigma)}(\mathcal{D}) \subset \mathcal{O}$. Conversely, consider $B \in \mathcal{O}$, then for any $D \in \mathcal{D}$, we have $BD = DB + D' \in \mathcal{N}$ with D' some element of $\mathcal{D}$, where we used $\mathcal{R}(X,\sigma)\mathcal{D} = \mathcal{R}(X,\sigma)(\mathcal{N}\cap\mathcal{N}^*) \subset \mathcal{N}$. Similarly we see that $DB \in \mathcal{N}^*$. But then $\mathcal{N} \ni BD = DB + D' \in \mathcal{N}^*$, so $BD \in \mathcal{N}\cap\mathcal{N}^* = \mathcal{D}$. Likewise $DB \in \mathcal{D}$ and so $B \in M_{\mathcal{R}(X,\sigma)}(\mathcal{D})$.

(iii) Since $\mathcal{C} \subset \mathcal{D}$ we see from the definition of $\mathcal{O}$ that $F \in \mathcal{O}$ implies that $[F, \mathcal{C}] \subset \mathcal{D}$. Conversely, let $[F, \mathcal{C}] \subset \mathcal{D}$ for some $F \in \mathcal{R}(X,\sigma)$. Now $F[\mathcal{R}(X,\sigma)\mathcal{C}] \subset [\mathcal{R}(X,\sigma)\mathcal{C}]$ and $F[\mathcal{C}\mathcal{R}(X,\sigma)] = [F\mathcal{C}\mathcal{R}(X,\sigma)] \subset [(\mathcal{C}\mathcal{R}(X,\sigma)+\mathcal{D})\mathcal{R}(X,\sigma)] \subset [\mathcal{C}\mathcal{R}(X,\sigma)]$ because $\mathcal{C}F + \mathcal{D} \subset \mathcal{C}F + \mathcal{N}^* \subset [\mathcal{C}\mathcal{R}(X,\sigma)]$. Thus $F\mathcal{D} = F([\mathcal{R}(X,\sigma)\mathcal{C}] \cap [\mathcal{C}\mathcal{R}(X,\sigma)]) \subset [\mathcal{R}(X,\sigma)\mathcal{C}] \cap [\mathcal{C}\mathcal{R}(X,\sigma)] = \mathcal{D}$. Similarly $\mathcal{D}F \subset \mathcal{D}$, and thus by (ii) we see $F \in \mathcal{O}$.

(iv) $\mathcal{D} \subset \mathcal{O}$ so by (i) it is the unique maximal C*–algebra annihilated by all the states $\omega \in \mathfrak{S}_D(\mathcal{O}) = \mathfrak{S}_D \restriction \mathcal{O}$ (since $\mathcal{C} \subset \mathcal{O}$). Thus $\mathcal{D} = [\mathcal{O}\mathcal{C}] \cap [\mathcal{C}\mathcal{O}]$. But $\mathcal{C} \subset \mathcal{D}$, so by (ii) $[\mathcal{O}\mathcal{C}] \subset \mathcal{D} \subset [\mathcal{O}\mathcal{C}]$ and so $\mathcal{D} = [\mathcal{O}\mathcal{C}] = [\mathcal{C}\mathcal{O}]$.

11 Appendix: Symplectic Spaces

We collect here some basic facts for symplectic spaces which are required for the preceding proofs. In this section X will be a real linear space with a nondegenerate symplectic form $\sigma : X \times X \to \mathbb{R}$, and for any subspace $S \subset X$ its symplectic complement will be denoted by $S^\perp := \{ f \in X \mid \sigma(f, S) = 0 \}$. By $X = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ we will mean that all S_i are nondegenerate and $S_i \subset S_j^\perp$ if $i \neq j$, and each $f \in X$ has a unique decomposition $f = f_1 + f_2 + \cdots + f_n$ such that $f_i \in S_i$ for all i.

11.1 Lemma *(i) If X is countably dimensional, then it has a symplectic basis, i.e. a basis $\{q_1, p_1; q_2, p_2; \ldots\}$ such that $\sigma(p_i, q_j) = \delta_{ij}$ and $0 = \sigma(q_i, q_j) = \sigma(p_i, p_j)$ for all i, j.*

(ii) For any symplectic space X we have that if S is a nondegenerate finite–dimensional subspace, then $X = S \oplus S^\perp$

(iii) For any symplectic space X and a finite linearly independent subset $\{q_1, q_2, \ldots, q_k\} \subset X$ such that $\sigma(q_i, q_j) = 0$ for all i, j, there is a set $\{p_1, p_2, \ldots, p_k\} \subset X$ such that $B := \{q_1, p_1; q_2, p_2; \ldots; q_k, p_k\}$ is a symplectic basis for $\mathrm{Span}(B)$.

Proof: (i) Let $(e_n)_{n\in\mathbb{N}}$ be a linear basis of X. We construct the basis elements p_n, q_n inductively as follows. If $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$ are already chosen, pick a minimal m with $e_m \notin \mathrm{Span}\{p_1, \ldots, p_k, q_1, \ldots, q_k\}$ and put

$$p_{k+1} := e_m - \sum_{i=1}^{k} \left(\sigma(e_m, q_i)p_i + \sigma(p_i, e_m)q_i\right)$$

to ensure that this element is σ-orthogonal to all previous ones. Then pick l minimal, such that $\sigma(p_{k+1}, e_l) \neq 0$, put

$$\tilde{q}_{k+1} := e_l - \sum_{i=1}^{k} (\sigma(e_l, q_i)p_i + \sigma(p_i, e_l)q_i)$$

and pick $q_{k+1} \in \mathbb{R}\tilde{q}_{k+1}$ with $\sigma(p_{k+1}, q_{k+1}) = 1$. This process can be repeated *ad infinitum* and produces the required basis of X because for each k, the span of $\{p_1, \ldots, p_k,\, q_1, \ldots, q_k\}$ contains at least $\{e_1, \ldots, e_k\}$.

(ii) Since S is finite dimensional and nondegenerate, we can choose by (i) a symplectic basis $\{q_1,\, p_1;\, q_2,\, p_2; \ldots;\, q_k,\, p_k\}$ for it. Given any $v \in X$ then

$$v_S := \sum_{i=1}^{k} (\sigma(v, q_i)\, p_i + \sigma(p_i, v)\, q_i) \in S$$

and $v - v_S \in S^\perp$, *i.e.* $\sigma(v - v_S, S) = 0$. Thus $X = \operatorname{Span}\{S \cup S^\perp\}$, and as σ is nondegenerate $S \cap S^\perp = \{0\}$. Moreover, if $0 = v + w$ where $v \in S$ and $w \in S^\perp$, then $v = -w \in S \cap S^\perp = \{0\}$, and hence any decomposition of an $x \in X$ as $x = x_1 + x_2$ where $x_1 \in S$, $x_2 \in S^\perp$ is unique. Thus $X = S \oplus S^\perp$.

(iii) We first find via the method of part (i), symplectic pairs $\{\widetilde{q}_1, r_1; \ldots; \widetilde{q}_k,\, r_k\} \subset X$ such that the nondegenerate subspaces $S_j := \operatorname{Span}\{\widetilde{q}_1, r_1; \ldots; \widetilde{q}_j,\, r_j\} \supset \{q_1, \ldots, q_j\}$ but $q_{j+1} \notin S_j$. We construct the basis elements $\widetilde{q}_i$, r_i inductively as follows. If $r_1, \ldots, r_j$ and $\widetilde{q}_1, \ldots, \widetilde{q}_j$ are already chosen, put

$$\widetilde{q}_{j+1} := q_{j+1} - \sum_{i=1}^{k} (\sigma(q_{j+1}, \widetilde{q}_i)r_i + \sigma(r_i, q_{j+1})\widetilde{q}_i)$$

to ensure that $\widetilde{q}_{j+1} \in S_j^\perp$. By (ii), $X = S_j \oplus S_j^\perp$ hence $S_j^\perp$ is nondegenerate, so there is an element $r_{j+1} \in S_j^\perp$ such that $\sigma(r_{j+1}, \widetilde{q}_{j+1}) = 1$. It follows that $q_{j+2} \notin S_{j+1}$ and that $\{q_1, \ldots, q_{j+1}\} \subset S_{j+1}$. This process can be repeated to produce the required symplectic bases. Next, we want to show that in S_k we can choose $\{p_1,\, p_2, \ldots,\, p_k\}$ such that $\{q_1,\, p_1;\, q_2,\, p_2; \ldots;\, q_k,\, p_k\}$ is a symplectic basis for S_k. Now $\{q_1, \ldots, q_k\} \subset \{q_1, \ldots, q_k\}^\perp$ where henceforth the symplectic complements are all taken in S_k. We claim that the containment $\{q_2, \ldots, q_k\}^\perp \supset \{q_1,\, q_2, \ldots, q_k\}^\perp$ is proper. The map $\varphi : S_k \to S_k^*$ by $\varphi_x(y) := \sigma(x, y)$ is a linear isomorphism by nondegeneracy of σ. Then for any set $R \subset S_k$ we have $\varphi(R^\perp) = R^0$ *i.e.* the annihilator of R in S_k^*, hence $\dim(R^\perp) = \dim(R^0) = 2k - \dim(\operatorname{Span}(R))$. Thus $\dim\{q_1, \ldots, q_j\}^\perp = 2k - j$ from which the claim follows. Thus there is an $r \in \{q_2, \ldots, q_k\}^\perp \backslash \{q_1,\, q_2, \ldots, q_k\}^\perp$ such that $\sigma(r, q_1) \neq 0$. In particular, let p_1 be that multiple of r such that $\sigma(p_1, q_1) = 1$. Let $T_1 := \operatorname{Span}\{q_1, p_1\}$ then $\{q_2, \ldots, q_k\} \subset T_1^\perp$, and by (ii) we have $S_k = T_1 \oplus T_1^\perp$ where $T_1^\perp$ is nondegenerate. Thus we can now repeat this procedure in $T_1^\perp$ starting from q_2 to obtain p_2. This procedure will exhaust S_k to produce the desired symplectic basis $\{q_1,\, p_1;\, q_2,\, p_2; \ldots;\, q_k,\, p_k\}$. ■

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